Physics PhD general examinations

1984-1986

Massachusetts Institute of Technology

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Massachusetts Institute of Technology

Department of Physics

October 2, 1984

Doctoral General Examination

Part I

Five Hours

Instructions

This examination is divided into four groups of 40 points each. USE A SEPARATE EXAMINATION BOOKLET FOR EACH GROUP. The number of points for each problem is indicated in brackets beside the problem. Be sure to list the problem number with each solution.

A diagram or sketch as part of the answer is often useful, particularly when problem calls for a qualitative response.

NO books or reference materials may be used. You may find some of the information on the next pages of help in your work.

Constant	Symbol		MKS	cgs
Boltzmann constant	k	=	1.38×10^{-23} J/^{o}K	1.38×10^{-16} erg/^{o}K
Electron mass	m_e	=	9.1×10^{-31} kg	9.1×10^{-28} g
Proton mass	m_p	=	1.67×10^{-27} kdg	1.67×10^{-24} g
Gravitational constant	G	=	6.67×10^{-11} N-m^2/kg^2	6.67×10^{-8} dyne-cm^2/g^2
Planck's constant	$h/2\pi$	=	1.05×10^{-34} J-s $h/2\pi=6.58\times10^{22}$ MeV sec	1.05×10^{-27} erg-sec
Speed of light	c	=	3.0×10^{8} m/sec $hc/2\pi=1.97\times10^{-11}$ MeV cm	3.0×10^{10} cm/sec
Electronic charge	e	=	1.6×10^{-19} coul	4.8×10^{-10} esu
Energy units	1 eV	=	1.6×10^{-19} J	$1/6\times10^{-12}$ erg
Permittivity of vacuum	ϵ_0	=	8.85×10^{-12} farad/m	
Permeability of vacuum	μ_0	=	1.26×10^{-6} henry/m	
Acceleration of gravity	g	=	9.8 m/sec^2	980 cm/sec^2
Stefan-Boltzmann constant	σ	=	5.7×10^{-8} Wm-^{2}K^{-4}	5.7×10^{-5} erg s^{-1}cm^{-2}K^{-4}
Binding energy of electron in hydrogen ground state		=	13.6 eV	
Bohr magneton	μ_B	=	$e(h/2\pi)/(2mc)=9.27\times10^{-24}$ joules/tesla $=9.27\times10^{-21}$ erg/gauss	
1 atmosphere of pressure		=	1.013×10^{5} newton/m^2	1.013×10^{6} dynes/cm
Avogadro's number		=		6.025×10^{23} /mole

INTEGRALS

$\int \sin x \, dx = -\cos x$

$\int \cos x \, dx = \sin x$

$\int x \sin x \, dx = \sin x - x \cos x$

$\int x \cos x \, dx = \cos x + x \sin x$

$\int \sinh x \, dx = \cosh x$

$\int \cosh x \, dx = \sinh x$

DERIVATIVES

$\frac{d}{dx} \sinh x = \cosh x$

$\frac{d}{dx} \cosh x = \sinh x$

$\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$

$\frac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \tanh x$

$\frac{d}{dx} \operatorname{csch} x = - \operatorname{csch} x \coth x$

$\frac{d}{dx} \coth x = - \operatorname{csch}^2 x$

DEFINITE INTEGRALS

$$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

$$\int_{-\infty}^\infty x^{2n} e^{-x^2/2a^2} dx = (2\pi a^2)^{1/2} \, a^{2n} \times 1\cdot3\cdot5\cdots(2n-1)$$

SERIES

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)x^2}{2!} \pm \cdots \quad \text{for } x^2 < 1$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{for } |x| \leq 1$$

$$\log(N!) = N \ln N - N + \cdots\cdots$$

FORMULAS OF VECTOR ANALYSIS

	Rectangular coordinates	Cylindrical coordinates	Spherical coordinates
Conversion to rectangular coordinates		$x = r\cos\varphi \quad y = r\sin\varphi \quad z = z$	$x = r\cos\varphi\sin\theta \quad y = r\sin\varphi\sin\theta \quad z = r\cos\theta$
Gradient	$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$	$\nabla\phi = \frac{\partial\phi}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial\phi}{\partial\varphi}\boldsymbol{\phi} + \frac{\partial\phi}{\partial z}\mathbf{k}$	$\nabla\phi = \frac{\partial\phi}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\boldsymbol{\theta} + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\boldsymbol{\phi}$
Divergence	$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\nabla\cdot\mathbf{A} = \frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}$	$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(A_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$
Curl	$\nabla\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$	$\nabla\times\mathbf{A} = \begin{vmatrix} \frac{1}{r}\mathbf{r} & \boldsymbol{\phi} & \frac{1}{r}\mathbf{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ A_r & rA_\varphi & A_z \end{vmatrix}$	$\nabla\times\mathbf{A} = \begin{vmatrix} \frac{\mathbf{r}}{r^2\sin\theta} & \frac{\boldsymbol{\theta}}{r\sin\theta} & \frac{\boldsymbol{\phi}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ A_r & rA_\theta & rA_\varphi\sin\theta \end{vmatrix}$
Laplacian	$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$	$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2}$	$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\nabla\times\nabla\psi = 0$$

$$\nabla\cdot(\nabla\times\mathbf{a}) = 0$$

$$\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$$

$$\nabla\cdot(\psi\mathbf{a}) = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$$

$$\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a}$$

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$

$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$$

$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}$$

If $\mathbf{x}$ is the coordinate of a point with respect to some origin, with magnitude $r = |\mathbf{x}|$, and $\mathbf{n} = \mathbf{x}/r$ is a unit radial vector, then

$$\nabla\cdot\mathbf{x} = 3 \qquad \nabla\times\mathbf{x} = 0$$

$$\nabla\cdot\mathbf{n} = \frac{2}{r} \qquad \nabla\times\mathbf{n} = 0$$

$$(\mathbf{a}\cdot\nabla)\mathbf{n} = \frac{1}{r}[\mathbf{a} - \mathbf{n}(\mathbf{a}\cdot\mathbf{n})] = \frac{\mathbf{a}_\perp}{r}$$

CLEBSCH-GORDAN COEFFICIENTS

$j_1 = \frac{1}{2} \qquad j_2 = \frac{1}{2}$:

		1	1	0	1
		1	0	0	-1
$\frac{1}{2}$	$\frac{1}{2}$	1			
$\frac{1}{2}$	$-\frac{1}{2}$		$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	
$-\frac{1}{2}$	$\frac{1}{2}$		$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$	
$-\frac{1}{2}$	$-\frac{1}{2}$				1

$j_1 = 1 \qquad j_2 = \frac{1}{2}$:

		$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
		$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$
1	$\frac{1}{2}$	1					
1	$-\frac{1}{2}$		$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{3}}$			
0	$\frac{1}{2}$		$\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{1}{3}}$			
0	$-\frac{1}{2}$				$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{3}}$	
-1	$\frac{1}{2}$				$\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{2}{3}}$	
-1	$-\frac{1}{2}$						1

$j_1 = 1 \qquad j_2 = 1$:

		2	2	1	2	1	0	2	1	2
		2	1	1	0	0	0	-1	-1	-2
1	1	1								
1	0		$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$						
0	1		$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$						
1	-1				$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{3}}$			
0	0				$\sqrt{\frac{2}{3}}$	0	$-\sqrt{\frac{1}{3}}$			
-1	1				$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{3}}$			
0	-1							$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	
-1	0							$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$	
-1	-1									1

d FUNCTIONS

$$d^j_{m'm}(\beta) \equiv \langle jm' | \exp(-i\beta J_y/\hbar) | jm \rangle$$

$j = \frac{1}{2}$:

$m' \backslash m$	$+\frac{1}{2}$	$-\frac{1}{2}$
$+\frac{1}{2}$	$\cos\beta/2$	$-\sin\beta/2$
$-\frac{1}{2}$	$\sin\beta/2$	$\cos\beta/2$

$j = 1$:

$m' \backslash m$	$+1$	0	-1
$+1$	$\frac{1}{2}(1+\cos\beta)$	$-\frac{1}{\sqrt{2}}\sin\beta$	$\frac{1}{2}(1-\cos\beta)$
0	$\frac{1}{\sqrt{2}}\sin\beta$	$\cos\beta$	$-\frac{1}{\sqrt{2}}\sin\beta$
-1	$\frac{1}{2}(1-\cos\beta)$	$\frac{1}{\sqrt{2}}\sin\beta$	$\frac{1}{2}(1+\cos\beta)$

MORE FORMULAS

$$L_\pm |\ell, m_\ell\rangle = \sqrt{\ell(\ell+1) - m_\ell(m_\ell \pm 1)}\,\hbar\,|\ell, m_\ell \pm 1\rangle$$

$$S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad S_y = \frac{\hbar}{2i}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \qquad S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad \text{Spin } \tfrac{1}{2} \text{ matrices}$$

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} \qquad S_y = \frac{\hbar}{\sqrt{2}i}\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 1\\ 0 & -1 & 0\end{pmatrix} \qquad S_z = \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix} \qquad \text{Spin 1 matrices}$$

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) - \frac{\hat{L}^2(\theta,\phi)\psi}{\hbar^2 r^2} \qquad \text{Spherical coordinates}$$

$$E = -\vec{\mu}\cdot\vec{B}$$

$$\vec{\mu} = -g_L \mu_B \vec{J}/\hbar \qquad g_L = 1 + \frac{\hat{\vec{S}}\cdot\hat{\vec{J}}}{J^2}$$

I.1 (10 points)

Motion in a Potential Well

A particle of mass m moves in a one dimensional potential $U(x) = A|x|^n$ where A is a constant. Find the dependence of the period τ on the energy E in terms of n.

I.2 (10 points)

Ideal Gas

An ideal gas is separated into two volumes, V_1 and V_2, by a piston such that each volume contains N atoms, and is at temperature T_0. The piston is manipulated reversibly until the gas on one side is in equilibrium with the other. The piston is then removed and the gas is allowed to mix. Assume that the piston is thermally conducting and that the whole system is thermally isolated from the outside world throughout. What is the final temperature and how much work is extracted from the system? The free energy of an ideal gas is

$$F = -NkT\left(1 + \ln\left(\frac{V}{N\lambda^3}\right)\right) \quad \text{where} \quad \lambda = \left(\frac{2\pi\hbar^2}{mkT}\right)^{1/2}$$

I.3 (10 points)

Current in a Wire

A straight metal wire with a circular cross-section of conductivity σ and cross sectional area A carries a steady current.

a) Determine the direction and magnitude of the Poynting vector at the surface of the wire.

b) Integrate the normal component of the Poynting vector over the surface of the wire for a segment of length L, and compare the result with the Joule heat produced in this segment.

I.4 (10 points)

Relativistic Correction

Find the correction term to the non-relativistic Hamiltonian, H, for the hydrogen atom, which represents, in first approximation, the effect of the relativistic correction to the kinetic energy.

Write an expression (no need to carry out integrals) for the shift in the ground state energy.

Which states (n, ℓ, m_ℓ) of the atom would be most altered by inclusion of this term in H?

Is the ground state energy raised or lowered?

II.1 (10 points)

Kepler's Second Law

Masses, m_1 and m_2, with position vectors, $\vec{r}_1$ and $\vec{r}_2$, are acted upon by a central force $\vec{F}(|\vec{r}|)$ where $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$. Show that $\vec{r}$ sweeps out equal areas in equal times (Kepler's Second Law).

II.2 (15 points)

Bose Condensation

Consider an ideal Bose gas of N particles of mass m and spin zero in a volume V and temperature T.

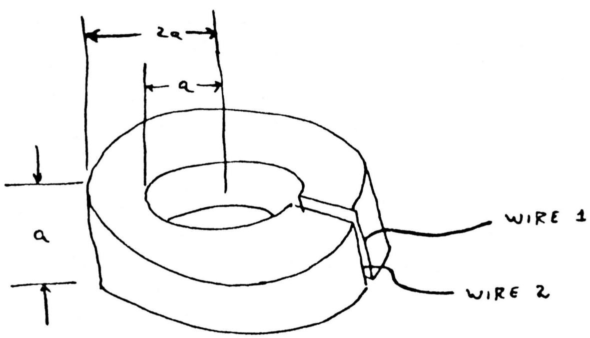
(i) What is the critical volume V_c below which Bose-Einstein condensation occurs? An answer up to a numerical constant will be sufficient.

(ii) What is the answer to part (i) in two dimensions? Explain your answer.

II.3 (5 points)

Resistance of a Washer

A washer is made of a dielectric of resistivity ρ. It has a square cross section of length a on a side and its outer radius is 2a. A small slit is made on one side and wires of negligible resistance are connected to the faces exposed by the slit. If the wires were connected into a circuit, what would be the lumped resistance due to the washer?



II.4 (10 points)

Bound States in a Square Well

A particle moves in a one-dimensional square well of width a and depth V. $V(x) = 0$ elsewhere.

How many bound states are there in this system as

(i) a is constant and $V \to 0$?

(ii) If V is constant and $a \to 0$?

Justify your answer.

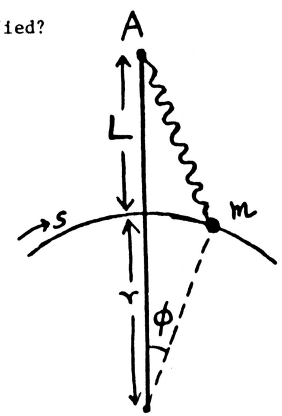
III.1 (15 points)

Spring and Hoop

Consider a particle of mass m which moves on a circular hoop of radius r and is attached to a spring with one end fixed at a point A at a distance L from the hoop, and in the plane of the hoop. Assume that a force G is required to extend the spring to length L, and that only small amplitude oscillations are considered so that $\phi \ll 1$ rad.

a) Write down the Lagrangian for the system and determine the frequency of oscillation assuming that the motion of mass m is frictionless.

b) Consider a frictional force $F = -\alpha\dot{s}$ acting between the mass m and the hoop. How is the motion modified?



III.2 (10points)

Digital Watch

A digital quartz clock loses time at a rate <0.1 sec./week in an office at 20°C. The clock is installed in a car which comes to thermal equilibrium at -5°C in an M.I.T. parking lot during the month of January. What is the loss rate of the clock now? The thermal expansion coefficient of quartz is $\alpha = 4.5 \times 10^{-7}$/°C. (You may assume that the elastic constant of quartz is independent of temperature.)

III.3 (5 points)

Refraction

Consider a potential that exists in the half-space, i.e.

$$V = 0 \qquad x < 0$$

$$V = -V_0 \qquad x > 0$$

Show that the classical trajectory of a particle with energy E approaching the potential from the left is identical with the refraction of light rays by a relative index of refraction

$$n = \sqrt{(E + V_o)/E}$$

III.4 (10 points)

Angular Momentum

In three dimensions, a particle mass m is subject to the potential

$$V(\vec{r}) = \begin{cases} -V_0 & |\vec{r}| < a \\ 0 & |\vec{r}| > a \end{cases}$$

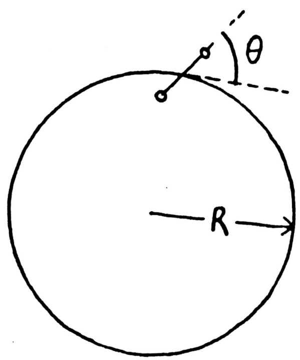
where $V_0 = 4 \dfrac{\hbar^2}{ma^2}$.

Is there an upper bound to the angular momentum of the bound states? Justify your answer.

IV.1 (10 points)

Spinning Dumbbell

Two spherical balls of mass m each and radius a fixed at the ends of a massless rod of length ℓ make up a dumbbell. Assume $a \ll \ell \ll R$. This object is in circular orbit about the earth a distance R from the earth's center. The dumbbell lies on the plane of the orbit and we denote the instantaneous orientation of the dumbbell relative to the orbital motion by the angle Θ. Find the angular acceleration of the dumbbell about its center of mass in terms of Θ.



IV.2 (10 points)

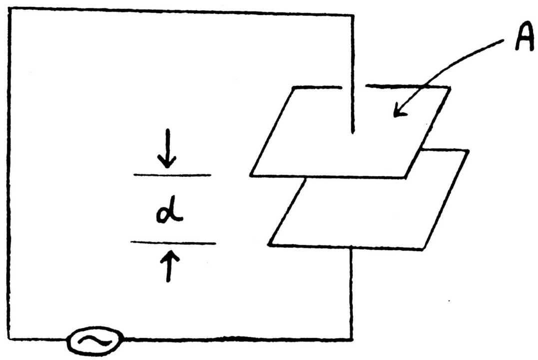
Electron occupation number

Consider a collection of atoms that are spatially separated from each other so that they do not interact. Each atom consists of a single bound state at an energy which varies from one atom to another. Denote the energy of the bound state of the ith atom by ε_i. Suppose the system is in equilibrium with a reservoir of electrons with chemical potential μ and temperature T. If we assume that the bound state is occupied by non-interacting electrons, we know that the average electron occupation number of the ith atom is given by $2f(\varepsilon_i)$ where $f(\varepsilon_i) = \frac{1}{1+e^{(\varepsilon_i-\mu)/kT}}$ is the Fermi function, and the factor of two accounts for up and down spins. However, due to Coulomb repulsion, it is more realistic to assume that each bound state may be occupied by a single electron with up or down spin, or unoccupied, but not doubly occupied. What is the average electron occupation of the ith atom in this case? Justify your answer.

IV.3 (10 points)

Radiating Capacitor

An oscillating voltage $V(t) = V_o \cos \omega t$ is applied across the leads to a plane parallel capacitor of plate area A and plate separation d. What is the lowest order radiation emitted? What is its time averaged intensity (in terms of V_o, d, A and other appropriate constants)?



$V = V_o \cos \omega t$

IV.4 (10 points)

Photon Scattering

A beam of photons of energy E is incident on a target of atomic hydrogen. The energy E is raised from E * 0 to 100 GeV. Describe the processes that can occur as a function of E, giving (qualitatively) Kinematics, thresholds and description of phenomena.

Massachusetts Institute of Technology

Department of Physics

October 4, 1984

Doctoral General Examination

Part II

Five Hours

Instructions

This examination consists of four sections with two problems each. Solve one problem from EACH OF THE FOUR SECTIONS (a total of four problems). It is advisable to read carefully both problems in each section before making your choice. The problems in each section have been chosen to be of comparable difficulty. USE A SEPARATE EXAMINATION BOOKLET FOR EACH PROBLEM AND LABEL IT WITH THE PROBLEM NUMBER AND YOUR NAME.

NO books or references may be used.

You may find some of the information on the next pages useful.

Constant	Symbol		MKS	cgs
Boltzmann constant	k	=	$1.38\text{x}10^{-23}$ J/^{o}K	$1.38\text{x}10^{-16}$ erg/^{o}K
Electron mass	m_e	=	$9.1\text{x}10^{-31}$ kg	$9.1\text{x}10^{-28}$ g
Proton mass	m_p	=	$1.67\text{x}10^{-27}$ kdg	$1.67\text{x}10^{-24}$ g
Gravitational constant	G	=	$6.67\text{x}10^{-11}$ N-m^2/kg^2	$6.67\text{x}10^{-8}$ dyne-cm^2/g^2
Planck's constant	$h/2\pi$	=	$1.05\text{x}10^{-34}$ J-s $h/2\pi$=$6.58\text{x}10^{22}$ MeV sec	$1.05\text{x}10^{-27}$ erg-sec
Speed of light	c	=	$3.0\text{x}10^{8}$ m/sec $hc/2\pi$=$1.97\text{x}10^{-11}$ MeV cm	$3.0\text{x}10^{10}$ cm/sec
Electronic charge	e	=	$1.6\text{x}10^{-19}$ coul	$4.8\text{x}10^{-10}$ esu
Energy units	1 eV	=	$1.6\text{x}10^{-19}$ J	$1/6\text{x}10^{-12}$ erg
Permittivity of vacuum	ϵ_0	=	$8.85\text{x}10^{-12}$ farad/m	
Permeability of vacuum	μ_0	=	$1.26\text{x}10^{-6}$ henry/m	
Acceleration of gravity	g	=	9.8 m/sec^2	980 cm/sec^2
Stefan-Boltzmann constant	σ	=	$5.7\text{x}10^{-8}$ Wm-^{2}K^{-4}	$5.7\text{x}10^{-5}$ erg s^{-1}cm^{-2}K^{-4}
Binding energy of electron in hydrogen ground state		=	13.6 eV	
Bohr magneton	μ_B	=	$e(h/2\pi)/(2mc)$=$9.27\text{x}10^{-24}$ joules/tesla =$9.27\text{x}10^{-21}$ erg/gauss	
1 atmosphere of pressure		=	1.013×10^5 newton/m^2	1.013×10^6 dynes/cm^2
Avogadro's number		=		6.025×10^{23} /mole

INTEGRALS

$\int \sin x \, dx = -\cos x$

$\int \cos x \, dx = \sin x$

$\int x \sin x \, dx = \sin x - x \cos x$

$\int x \cos x \, dx = \cos x + x \sin x$

$\int \sinh x \, dx = \cosh x$

$\int \cosh x \, dx = \sinh x$

DERIVATIVES

$\frac{d}{dx} \sinh x = \cosh x$

$\frac{d}{dx} \cosh x = \sinh x$

$\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$

$\frac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \tanh x$

$\frac{d}{dx} \operatorname{csch} x = -\operatorname{csch} x \coth x$

$\frac{d}{dx} \coth x = -\operatorname{csch}^2 x$

DEFINITE INTEGRALS

$$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

$$\int_{-\infty}^{\infty} x^{2n} e^{-x^2/2a^2} dx = (2\pi a^2)^{1/2} \, a^{2n} \times 1 \cdot 3 \cdot 5 \cdots (2n-1)$$

SERIES

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)x^2}{2!} \pm \cdots \quad \text{for } x^2 < 1$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{for } |x| \leq 1$$

$$\log(N!) = N \ln N - N + \ldots\ldots$$

FORMULAS OF VECTOR ANALYSIS

	Rectangular coordinates	Cylindrical coordinates	Spherical coordinates
Conversion to rectangular coordinates		$x = r\cos\varphi \quad y = r\sin\varphi \quad z = z$	$x = r\cos\varphi\sin\theta \quad y = r\sin\varphi\sin\theta \quad z = r\cos\theta$
Gradient............	$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$	$\nabla\phi = \frac{\partial\phi}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial\phi}{\partial\varphi}\boldsymbol{\phi} + \frac{\partial\phi}{\partial z}\mathbf{k}$	$\nabla\phi = \frac{\partial\phi}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\boldsymbol{\theta} + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\boldsymbol{\phi}$
Divergence..........	$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\nabla\cdot\mathbf{A} = \frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}$	$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(A_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$
Curl................	$\nabla\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$	$\nabla\times\mathbf{A} = \begin{vmatrix} \frac{1}{r}\mathbf{r} & \boldsymbol{\phi} & \frac{1}{r}\mathbf{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ A_r & rA_\varphi & A_z \end{vmatrix}$	$\nabla\times\mathbf{A} = \begin{vmatrix} \frac{\mathbf{r}}{r^2\sin\theta} & \frac{\boldsymbol{\theta}}{r\sin\theta} & \frac{\boldsymbol{\phi}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ A_r & rA_\theta & rA_\varphi\sin\theta \end{vmatrix}$
Laplacian...........	$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$	$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2}$	$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\nabla\times\nabla\psi = 0$$

$$\nabla\cdot(\nabla\times\mathbf{a}) = 0$$

$$\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$$

$$\nabla\cdot(\psi\mathbf{a}) = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$$

$$\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a}$$

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$

$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$$

$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}$$

If $\mathbf{x}$ is the coordinate of a point with respect to some origin, with magnitude $r = |\mathbf{x}|$, and $\mathbf{n} = \mathbf{x}/r$ is a unit radial vector, then

$$\nabla\cdot\mathbf{x} = 3 \qquad \nabla\times\mathbf{x} = 0$$

$$\nabla\cdot\mathbf{n} = \frac{2}{r} \qquad \nabla\times\mathbf{n} = 0$$

$$(\mathbf{a}\cdot\nabla)\mathbf{n} = \frac{1}{r}[\mathbf{a} - \mathbf{n}(\mathbf{a}\cdot\mathbf{n})] = \frac{\mathbf{a}_\perp}{r}$$

CLEBSCH-GORDAN COEFFICIENTS

$j_1 = \frac{1}{2} \qquad j_2 = \frac{1}{2}$:

$$
\begin{array}{cc|cccc}
 & & 1 & 1 & 0 & 1 \\
 & & 1 & 0 & 0 & -1 \\
\hline
\frac{1}{2} & \frac{1}{2} & 1 & & & \\
\frac{1}{2} & -\frac{1}{2} & & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & \\
-\frac{1}{2} & \frac{1}{2} & & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & \\
-\frac{1}{2} & -\frac{1}{2} & & & & 1
\end{array}
$$

$j_1 = 1 \qquad j_2 = \frac{1}{2}$:

$$
\begin{array}{cc|cccccc}
 & & \frac{3}{2} & \frac{3}{2} & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & \frac{3}{2} \\
 & & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} \\
\hline
1 & \frac{1}{2} & 1 & & & & & \\
1 & -\frac{1}{2} & & \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & & & \\
0 & \frac{1}{2} & & \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{3}} & & & \\
0 & -\frac{1}{2} & & & & \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & \\
-1 & \frac{1}{2} & & & & \sqrt{\frac{1}{3}} & -\sqrt{\frac{2}{3}} & \\
-1 & -\frac{1}{2} & & & & & & 1
\end{array}
$$

$j_1 = 1 \qquad j_2 = 1$:

$$
\begin{array}{cc|ccccccccc}
 & & 2 & 2 & 1 & 2 & 1 & 0 & 2 & 1 & 2 \\
 & & 2 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -2 \\
\hline
1 & 1 & 1 & & & & & & & & \\
1 & 0 & & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & & & & & & \\
0 & 1 & & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & & & & & & \\
1 & -1 & & & & \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{3}} & & & \\
0 & 0 & & & & \sqrt{\frac{2}{3}} & 0 & -\sqrt{\frac{1}{3}} & & & \\
-1 & 1 & & & & \sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{3}} & & & \\
0 & -1 & & & & & & & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & \\
-1 & 0 & & & & & & & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & \\
-1 & -1 & & & & & & & & & 1
\end{array}
$$

d FUNCTIONS

$$d^j_{m'm}(\beta) \equiv \langle jm' | \exp(-i\beta J_y/\hbar) | jm \rangle$$

$j = \frac{1}{2}$:

m' \ m	$+\frac{1}{2}$	$-\frac{1}{2}$
$+\frac{1}{2}$	$\cos\beta/2$	$-\sin\beta/2$
$-\frac{1}{2}$	$\sin\beta/2$	$\cos\beta/2$

$j = 1$:

m' \ m	$+1$	0	-1
$+1$	$\frac{1}{2}(1+\cos\beta)$	$-\frac{1}{\sqrt{2}}\sin\beta$	$\frac{1}{2}(1-\cos\beta)$
0	$\frac{1}{\sqrt{2}}\sin\beta$	$\cos\beta$	$-\frac{1}{\sqrt{2}}\sin\beta$
-1	$\frac{1}{2}(1-\cos\beta)$	$\frac{1}{\sqrt{2}}\sin\beta$	$\frac{1}{2}(1+\cos\beta)$

MORE FORMULAS

$$L_\pm |\ell, m_\ell\rangle = \sqrt{\ell(\ell+1) - m_\ell(m_\ell \pm 1)}\,\hbar\,|\ell, m_\ell \pm 1\rangle$$

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y = \frac{\hbar}{2i}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{Spin } \tfrac{1}{2} \text{ matrices}$$

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad S_y = \frac{\hbar}{\sqrt{2}i}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad S_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \text{Spin 1 matrices}$$

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) - \frac{\hat{L}^2(\theta,\phi)\psi}{\hbar^2 r^2} \qquad \text{Spherical coordinates}$$

$$E = -\vec{\mu}\cdot\vec{B}$$

$$\vec{\mu} = -g_L \mu_B \vec{J}/\hbar \qquad g_L = 1 + \frac{\hat{\vec{S}}\cdot\hat{\vec{J}}}{J^2}$$

I.1 Adiabatic Invariants

Consider a harmonic oscillator consisting of a mass m attached to a spring with spring constant $k = m\omega^2$. The mass is constrained to move in one dimension only.

a) Write down the Hamiltonian in terms of the coordinates x (which locates the mass) and its corresponding momentum p. Solve for the motion using Hamilton's equations

$$\dot{x} = \frac{\delta H}{\delta p} \qquad \dot{p} = -\frac{\delta H}{\delta x}$$

Express x(t) and p(t) in terms of the oscillator energy E.

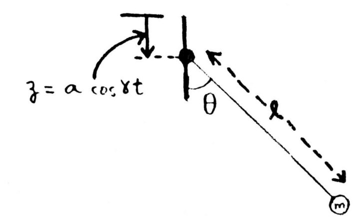
b) Using the results of part a) compute the action $I = \frac{1}{2\pi}\oint p\,dx$. Show that $I = E/\omega$.

c) Assume that the spring constant is a slowly varying function of time, i.e. $\frac{\dot{k}}{k} \ll \omega$. Show that $\left\langle \frac{dI}{dt} \right\rangle = 0$ to lowest order by taking appropriate averages over the period of the motion.

d) Suppose that over a long time the spring constant k has decreased by a factor of two. What is the change in the vibrational amplitude of the mass?

I.2 The Inverted Pendulum

Consider a simple pendulum whose point of support oscillates vertically according to $z(t) = a\cos\gamma t$. The pendulum consists of a mass m and a rigid, massless rod of length ℓ in a gravitational field of strength g. (See figure.) (At t=0, the point of support is at its maximum downward extension.)



a) Write down the Lagrangian in terms of the generalized coordinate θ shown in the figure.

b) Find the equation of motion using the Lagrangian of part a). Simplify your result by considering small oscillations about $\theta=\pi$, i.e. the pendulum is inverted. Show that the motion can be described by Mathieu's equation

$$\ddot{x} + A(1 + h\cos\beta t)x = 0 \,. \qquad \text{(I)}$$

(Hint: put $\phi = \pi-\theta$)

c) Assume that we can write the solution as the sum of a slowly varying function X and a rapidly varying function ξ with $\xi \ll X$. For specificity, take $h \ll 1$ so that ξ varies with frequency β. Substitute $x(t) = X + \xi$ into equation (I) and find appropriate equations for $X(t,\xi)$ and $\xi(t)$.

d) Solve for $\xi(t)$ and use this solution in the equation for X. Derive an equation for X by averaging the equation just found over the time period $\frac{2\pi}{\beta}$.

e) Find a lower limit on the driving frequency γ such that the inverted pendulum is stable with respect to small oscillations. Express your result in terms of g, ℓ, and a. (A more sophisticated analysis making use of the characteristics of Mathieu's equation shows that γ can also have an upper limit for some values of g, ℓ, and a.)

CONTINUED ON NEXT PAGE

Useful formulae:

$$\cos x \cos y = \frac{1}{2}\cos(x+y) + \frac{1}{2}\cos(x-y)$$

$$\sin x \cos y = \frac{1}{2}\sin(x+y) + \frac{1}{2}\sin(x-y)$$

II.1 Alfven Waves

Consider a conducting medium subject to a magnetic field B. The medium is described by an instantaneous density ρ_m, velocity $\vec{v}$, pressure P and conductivity $\sigma=1/\eta$. The equation of motion of the medium is as follows:

$$\frac{\partial \rho_m}{\partial t} + \vec{\nabla}\cdot(\rho_m \vec{v}) = 0 \qquad \text{(continuity)}$$

$$\rho_m \frac{d\vec{v}}{dt} = \frac{\vec{J}\times\vec{B}}{c} - \vec{\nabla}P \qquad \text{(momentum transport)}$$

$$\vec{E} + \frac{\vec{v}\times\vec{B}}{c} = \eta\vec{J} \qquad \text{(Ohm's law)}$$

$$P \propto \rho_m^{\gamma} \qquad \text{(Equation of state)} .$$

The medium is at rest and subject to a uniform magnetic field B_0. In addition to the ordinary sound wave, a new propagating wave appears (the Alfven wave). This is studied by considering small oscillations in the medium and in the magnetic field.

a) By combining the above with Maxwell's equations, show that the time variation of the magnetic field may be described by the following equation:

$$\frac{\partial\vec{B}}{\partial t} = \vec{\nabla}\times(\vec{v}\times\vec{B}) + \frac{c^2}{4\pi\sigma}\nabla^2\vec{B}$$

where $\sigma = 1/\eta$ is the conductivity (assumed to be constant). Assume $V_A^2 \ll c^2$, where V_A is the Alfven speed given by $V_A{}^2 = B_o{}^2/4\pi\rho_m$.

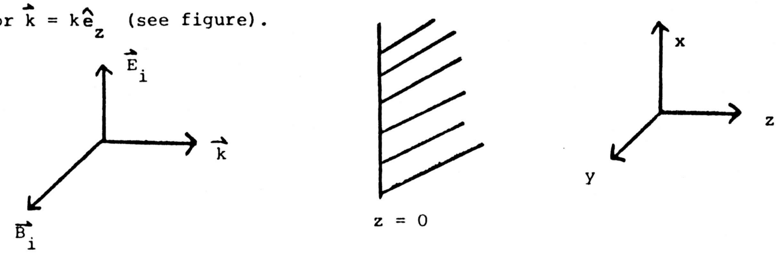
b) From the relationship given in a), find the dispersion relationship of MHD wave propagation parallel to the external dc magnetic field (shear Alfven waves), including dissipation. What is the spatial scale length of the damping of these perturbations in the limit of weak dissipation? Do the sound waves contribute to this dispersion relationship? What if the waves propagated at an angle to $\vec{B}$?

c) Consider the case $\sigma = \infty$. Evaluate the energy density, the Poynting flux, and the group velocity, and show that these are appropriately related.

d) Would this wave be reflected from an infinitely conducting rigid plane surface normal to the magnetic field? What if the conducting sheet were not rigid? Support your answers.

II.2 Radiation Pressure

A plane polarized electromagnetic wave of frequency ω propagates through a vacuum and strikes an infinitely thick conductor at normal incidence. The wave is described by an electric field $\vec{E}_i$, magnetic field $\vec{B}_i$ and wave vector $\vec{k} = k\hat{e}_z$ (see figure).



The conductor surface fills the xy plane and the conductor extends from $z = 0$ to $z \longrightarrow \infty$. The conductor has a conductivity σ, a permittivity ε, and a permeability $\mu = 1$; σ and ε are constants. We want to calculate the force per unit area exerted by the wave on the conductor, i.e. the radiation pressure in terms if the amplitude of the incident wave E_{io}.

a) Assume the wave varies as $e^{ikz-i\omega t}$. Use Maxwell's equations to show that the dispersion relation is

i) $k = \frac{\omega}{c}$ in vacuum

ii) $k^2 = \varepsilon \frac{\omega^2}{c^2} \left(1 + i\,\frac{4\pi\sigma}{\omega\varepsilon}\right)$ in the conductor. Assume the charge density in the conductor is always zero.

b) Suppose the electric field in the conductor can be written as

$$\vec{E}_c = \mathrm{Re}(E_{co}e^{ikz-i\omega t})\hat{e}_x$$

Calculate the instantaneous volume force exerted on the conductor in terms of $|E_c|$ and other appropriate constants. Assume we are dealing with an excellent conductor such that $\sigma \gg \omega$. Use this fact to simplify your algebra.

c) Using the result of part b) calculate the time averaged pressure exerted on the force of the conductor by the wave.

d) Use appropriate boundary conditions to relate $|E_{co}|$ to $|E_{io}|$. the amplitude of the incident wave.

CONTINUED ON NEXT PAGE

e) Using the results of parts c) and d) write down the radiation pressure in terms of $|E_{io}|$. How does the pressure vary with σ in this case?

III.1 Melting Curve of He^3

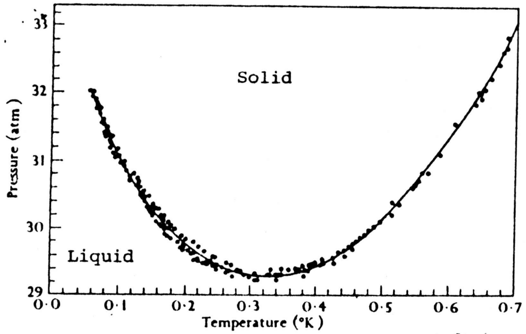


Figure 1

He^3 at low temperature is converted from liquid to solid by the application of pressure. The melting curve P(T) is shown in Fig. 1. Note the peculiar feature that dP/dT is negative at low temperature. We will attempt to understand the phase diagram by a simple model of the liquid and solid phases.

(i) The solid phase consists of He^3 atoms on a crystalline lattice. Each atom has nuclear spin of 1/2. Ignoring the interaction between the spins, what is the entropy per particle s_S due to the spin degrees of freedom?

(ii) The liquid phase can be modelled as an ideal Fermi gas. The volume per particle v_L of liquid He^3 is 46 $\mathring{A}^3$/atom. What is the Fermi temperature T_F in degrees Kelvin?

(iii) What is the entropy per particle s_L of the liquid at low temperature? You may give your answer up to a numerical constant. It is sufficient to provide a physical justification of your answer. For $T \ll T_F$, does solid or liquid He^3 have a higher entropy per particle?

(iv) Derive the Clapeyron equation which relates dP/dT with $\Delta s = s_L - s_S$ and $\Delta v = v_L - v_S$ where v_S and v_L are the volume epr particle of the solid and liquid respectively. Δv is measured to be 3 $\mathring{A}^3$/ atoms. Make an estimate for dP/dT and compare with the experiment shown in Fig. 1.

CONTINUED ON NEXT PAGE

(v) Describe what happens when additional pressure is applied to a thermally isolated mixture of solid and liquid He^3 in coexistence. (Hint: These considerations form the basis of a refrigerator proposed by I. Pomeranchuk in 1950.)

III.2 Vapor Pressure of a Solid

At a given temperature, a solid exists in equilibrium with its vapor at the pressure P(T). Our goal in this problem is to produce an expression for the vapor pressure P(T).

(i) We approximate the gas as an ideal gas with N atoms of mass m. Show that the free energy of a classical ideal gas is given by $F = -NkT\left(1 + \ln\left(\frac{V}{N\lambda^3}\right)\right)$ where

$$\lambda = \left(\frac{2\pi\hbar^2}{mkT}\right)^{1/2} .$$

(ii) In the solid phase, due to the interaction between the atoms, the ground state gains an energy $-\varepsilon_B$ per atom. We model the thermal excitation of the solid using the Einstein model in which each degree of freedom of the atoms is described by an oscillator at a frequency ω_o. What is the free energy of the solid within this approximation?

(iii) Derive the condition for equilibrium between the solid and gas. Using this condition, calculate the vapor pressure P(T).

(iv) Now consider a gas of diatomic molecules. It is reasonable to assume that the vibrational degrees of freedom is the same in the gas and solid phase. In the gas phase the molecule can rotate freely and you may take the classical limit in describing this rotation. In the solid phase the molecule is no longer free to rotate. Within the spirit of the present approximations, how is the expression for the vapor pressure modified?

IV.1 Spin Polarized Electrons

Consider an atom in which the highest filled state is a p state which is split by spin-orbit into $p_{3/2}$ and $p_{1/2}$. The p states are completely filled and the lowest empty state is an $s_{1/2}$ state. Right circularly polarized light at a frequency $\Omega = E_3 - E_2$ is incident on the atom.

$s_{1/2}$ ———————— E_3

$p_{3/2}$ ———————— E_2

$p_{1/2}$ ———————— E_1

(i) What are the degeneracies of the states? What are the allowed excitations?

(ii) Show that there is a net spin polarization of the electrons in the excited state. What is the direction of the polarization? Let $n_\uparrow$ and $n_\downarrow$ denote the probability of finding the atom **excited** to the s state with spin up and down. Calcula the relative polarization in the excited atom,

$$M = \frac{n_\uparrow - n_\downarrow}{n_\uparrow + n_\downarrow} .$$

It is worth noting that the excited electrons can be extracted to form an efficient polarized electron source.

IV.2 Virial Theorem

a) Consider N particles with coordinates q_i, $i=1\ldots3N$, subject to a potential $V(\{q_i\})$.

Prove the quantum mechanical virial theorem:

$$\langle\psi|\sum_i q_i \frac{\partial V}{\partial q_i}|\psi\rangle = 2\langle\psi|\hat{T}|\psi\rangle \quad ,$$

where $\hat{T}$ is the kinetic energy operator, and $|\psi\rangle$ is an eigenstate of H.

(Hint: the more familiar classical virial theorem is proven by considering the time dependence of the quantity $\sum_i q_i p_i$ where p_i is the momentum conjugate to q_i.)

b) Apply the virial theorem to evaluate the ratio of the kinetic to potential energy

$$R = \frac{\langle\psi|\hat{T}|\psi\rangle}{\langle\psi|V|\psi\rangle}$$

for (i) the harmonic oscillator

(ii) the hydrogen atom

(iii) an N electron atom.

c) What is the condition for the validity of your proof of the virial theorem?

Can you give a simple example of the breakdown of the virial theorem?

MIT

Massachusetts Institute of Technology

Department of Physics

February 12, 1985

Doctoral General Examination

Part I

Five Hours

Instructions

This examination is divided into four groups of 40 points each. USE A SEPARATE EXAMINATION BOOKLET FOR EACH GROUP. The number of points for each problem is indicated in brackets beside the problem. Be sure to list the problem number with each solution.

A diagram or sketch as part of the answer is often useful, particularly when problem calls for a qualitative response.

NO books or reference materials may be used. You may find some of the information on the next pages of help in your work.

MIT

Constant	Symbol		MKS	cgs
Boltzmann constant	k	=	$1.38\times10^{-23} J/^oK$	$1.38\times10^{-16} erg/^oK$
Electron mass	m_e	=	9.1×10^{-31} kg	9.1×10^{-28} g
Proton mass	m_p	=	1.67×10^{-27} kdg	1.67×10^{-24} g
Gravitational constant	G	=	6.67×10^{-11} $N\text{-}m^2/kg^2$	6.67×10^{-8} $dyne\text{-}cm^2/g^2$
Planck's constant	$h/2\pi$	=	1.05×10^{-34} J-s $h/2\pi=6.58\times10^{22}$ MeV sec	1.05×10^{-27} erg-sec
Speed of light	c	=	3.0×10^{8} m/sec $hc/2\pi=1.97\times10^{-11}$ MeV cm	3.0×10^{10} cm/sec
Electronic charge	e	=	1.6×10^{-19} coul	4.8×10^{-10} esu
Energy units	1 eV	=	1.6×10^{-19} J	$1/6\times10^{-12}$ erg
Permittivity of vacuum	ε_0	=	8.85×10^{-12} farad/m	
Permeability of vacuum	μ_0	=	1.26×10^{-6} henry/m	
Acceleration of gravity	g	=	9.8 m/sec^2	980 cm/sec^2
Stefan-Boltzmann constant	σ	=	5.7×10^{-8} $Wm^{-2}K^{-4}$	5.7×10^{-5} erg $s^{-1}cm^{-2}K^{-4}$
Binding energy of electron in hydrogen ground state		=	13.6 eV	
Bohr magneton	μ_B	=	$e(h/2\pi)/(2mc)=9.27\times10^{-24}$ joules/tesla $=9.27\times10^{-21}$ erg/gauss	
1 atmosphere of pressure		=	1.013×10^5 newton/m^2	1.013×10^6 dynes/cm^2
Avogadro's number		=		6.025×10^{23} /mole

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INTEGRALS

$\int \sin x \, dx = -\cos x$

$\int \cos x \, dx = \sin x$

$\int x \sin x \, dx = \sin x - x \cos x$

$\int x \cos x \, dx = \cos x + x \sin x$

$\int \sinh x \, dx = \cosh x$

$\int \cosh x \, dx = \sinh x$

DERIVATIVES

$\frac{d}{dx} \sinh x = \cosh x$

$\frac{d}{dx} \cosh x = \sinh x$

$\frac{d}{dx} \tanh x = \mathrm{sech}^2 x$

$\frac{d}{dx} \mathrm{sech}\, x = -\mathrm{sech}\, x \tanh x$

$\frac{d}{dx} \mathrm{csch}\, x = -\mathrm{csch}\, x \coth x$

$\frac{d}{dx} \coth x = -\mathrm{csch}^2 x$

DEFINITE INTEGRALS

$$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

$$\int_{-\infty}^\infty x^{2n} e^{-x^2/2a^2} dx = (2\pi a^2)^{1/2} \, a^{2n} \times 1 \cdot 3 \cdot 5 \cdots (2n-1)$$

SERIES

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)x^2}{2!} \pm \cdots \text{ for } x^2 < 1$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots$$

$$\log (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \text{ for } |x| \leq 1$$

$$\log (N!) = N \ln N - N + \ldots\ldots$$

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FORMULAS OF VECTOR ANALYSIS

	Rectangular coordinates	Cylindrical coordinates	Spherical coordinates
Conversion to rectangular coordinates		$x = r\cos\varphi \quad y = r\sin\varphi \quad z = z$	$x = r\cos\varphi\sin\theta \quad y = r\sin\varphi\sin\theta \quad z = r\cos\theta$
Gradient	$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$	$\nabla\phi = \frac{\partial\phi}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial\phi}{\partial\varphi}\boldsymbol{\phi} + \frac{\partial\phi}{\partial z}\mathbf{k}$	$\nabla\phi = \frac{\partial\phi}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\boldsymbol{\theta} + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\boldsymbol{\phi}$
Divergence	$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\nabla\cdot\mathbf{A} = \frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}$	$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(A_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$
Curl	$\nabla\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$	$\nabla\times\mathbf{A} = \begin{vmatrix} \frac{1}{r}\mathbf{r} & \boldsymbol{\phi} & \frac{1}{r}\mathbf{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ A_r & rA_\varphi & A_z \end{vmatrix}$	$\nabla\times\mathbf{A} = \begin{vmatrix} \frac{\mathbf{r}}{r^2\sin\theta} & \frac{\boldsymbol{\theta}}{r\sin\theta} & \frac{\boldsymbol{\phi}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ A_r & rA_\theta & rA_\varphi\sin\theta \end{vmatrix}$
Laplacian	$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$	$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2}$	$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\nabla\times\nabla\psi = 0$$

$$\nabla\cdot(\nabla\times\mathbf{a}) = 0$$

$$\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$$

$$\nabla\cdot(\psi\mathbf{a}) = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$$

$$\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a}$$

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$

$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$$

$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}$$

If $\mathbf{x}$ is the coordinate of a point with respect to some origin, with magnitude $r = \lvert\mathbf{x}\rvert$, and $\mathbf{n} = \mathbf{x}/r$ is a unit radial vector, then

$$\nabla\cdot\mathbf{x} = 3 \qquad \nabla\times\mathbf{x} = 0$$

$$\nabla\cdot\mathbf{n} = \frac{2}{r} \qquad \nabla\times\mathbf{n} = 0$$

$$(\mathbf{a}\cdot\nabla)\mathbf{n} = \frac{1}{r}[\mathbf{a} - \mathbf{n}(\mathbf{a}\cdot\mathbf{n})] = \frac{\mathbf{a}_\perp}{r}$$

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CLEBSCH-GORDAN COEFFICIENTS

$j_1 = \frac{1}{2} \qquad j_2 = \frac{1}{2}$:

$$\begin{array}{cc}
 & \begin{array}{cccc} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{array} \\
\begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{array} &
\begin{bmatrix} 1 & & & \\ & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & \\ & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & \\ & & & 1 \end{bmatrix}
\end{array}$$

$j_1 = 1 \qquad j_2 = \frac{1}{2}$:

$$\begin{array}{cc}
 & \begin{array}{cccccc} \frac{3}{2} & \frac{3}{2} & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} \end{array} \\
\begin{array}{cc} 1 & \frac{1}{2} \\ 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} \\ -1 & \frac{1}{2} \\ -1 & -\frac{1}{2} \end{array} &
\begin{bmatrix} 1 & & & & & \\ & \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & & & \\ & \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{3}} & & & \\ & & & \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & \\ & & & \sqrt{\frac{1}{3}} & -\sqrt{\frac{2}{3}} & \\ & & & & & 1 \end{bmatrix}
\end{array}$$

$j_1 = 1 \qquad j_2 = 1$:

$$\begin{array}{cc}
 & \begin{array}{ccccccccc} 2 & 2 & 1 & 2 & 1 & 0 & 2 & 1 & 2 \\ 2 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -2 \end{array} \\
\begin{array}{cc} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & -1 \\ 0 & 0 \\ -1 & 1 \\ 0 & -1 \\ -1 & 0 \\ -1 & -1 \end{array} &
\begin{bmatrix} 1 & & & & & & & & \\ & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & & & & & & \\ & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & & & & & & \\ & & & \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{3}} & & & \\ & & & \sqrt{\frac{2}{3}} & 0 & -\sqrt{\frac{1}{3}} & & & \\ & & & \sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{3}} & & & \\ & & & & & & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & \\ & & & & & & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & \\ & & & & & & & & 1 \end{bmatrix}
\end{array}$$

---

d FUNCTIONS

$$d^{j}_{m'm}(\beta) \equiv \langle jm' | \exp(-i\beta J_y/\hbar) | jm \rangle$$

$j = \frac{1}{2}$:

$m' \backslash m$	$+\frac{1}{2}$	$-\frac{1}{2}$
$+\frac{1}{2}$	$\cos \beta/2$	$-\sin \beta/2$
$-\frac{1}{2}$	$\sin \beta/2$	$\cos \beta/2$

$j = 1$:

$m' \backslash m$	$+1$	0	-1
$+1$	$\frac{1}{2}(1 + \cos \beta)$	$-\frac{1}{\sqrt{2}} \sin \beta$	$\frac{1}{2}(1 - \cos \beta)$
0	$\frac{1}{\sqrt{2}} \sin \beta$	$\cos \beta$	$-\frac{1}{\sqrt{2}} \sin \beta$
-1	$\frac{1}{2}(1 - \cos \beta)$	$\frac{1}{\sqrt{2}} \sin \beta$	$\frac{1}{2}(1 + \cos \beta)$

MORE FORMULAS

$$L_{\pm}|\ell,m_\ell> \; = \sqrt{\ell(\ell+1) - m_\ell\,(m_\ell \pm 1)}\,\hbar\,|\ell,m_\ell \pm 1>$$

$$S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad S_y = \frac{\hbar}{2i}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \qquad S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad \text{Spin } \tfrac{1}{2} \text{ matrices}$$

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} \qquad S_y = \frac{\hbar}{\sqrt{2}i}\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 1\\ 0 & -1 & 0\end{pmatrix} \qquad S_z = \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix} \qquad \text{Spin 1 matrices}$$

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) - \frac{\hat{L}^2(\theta,\phi)\psi}{\hbar^2 r^2} \qquad \text{Spherical coordinates}$$

$$E = -\vec{\mu}\cdot\vec{B}$$

$$\vec{\mu} = -g_L \mu_B\,\vec{J}/\hbar \qquad\qquad g_L = 1 + \frac{\hat{\vec{S}}\cdot\hat{\vec{J}}}{J^2}$$

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I.1 (10 points)

Square-Well

A particle is in an infinite one-dimensional square well of width a. At t = 0 the particle is in the left half of the well with a wavefunction $\Psi_A(x)$ where

$$\Psi_A(x) = \sqrt{\frac{2}{a}} \quad \text{-- left half of well}$$

$$= 0 \quad \text{--- right half of well}$$

1. Find the time dependent wavefunction $\Psi_A(x,t)$ for $t \gtrsim 0$.

2. Find the probability that the particle is in the n^{th} eigenstate.

3. Write an expression for the average value of the particle's energy.

I.2 (10 points)

Flowing Fluid

Consider a non-uniformly flowing fluid. The fluid below a plane with normal $\hat{z}$ exerts a force unit area $\vec{P}_z$ on the fluid above the plane. The viscosity coefficient η is defined by

$$\vec{P}_z = - \eta \partial \vec{u} / \partial z,$$

to leading order, where $\vec{u}$ is the local average fluid velocity.

For a dilute classical gas of molecules of mass m and scattering cross section σ, at temperature T, derive η. (You may ignore numerical factors).

I.3 (10 points)

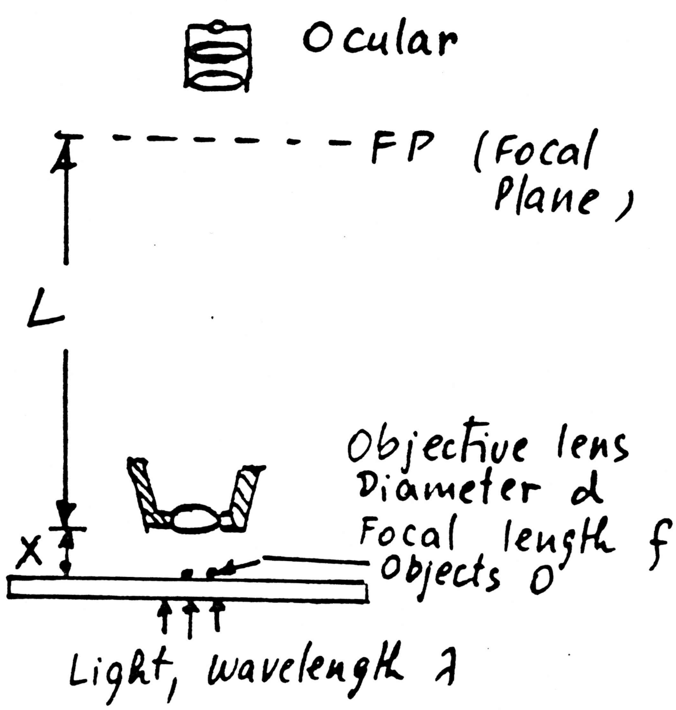
Microscope

Two very tiny objects at O, separated by ε, are to be inspected with a powerful microscope composed of the objective lens, focal length f, diameter d, and an ocular used to inspect the image formed at the focal plane FP. $L \gg X$. The microscope uses monochromatic light of wavelength λ:

1. Sketch the image at the focal plane FP when only one object -- assumed to be a point — is present, and give an approximate expression for the image size in terms of given constants.

2. Sketch the images formed when the two point-like objects are close together.

3. Assuming λ is constant give an expression for the spatial resolution of this microscope.



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I.4 (10 points)

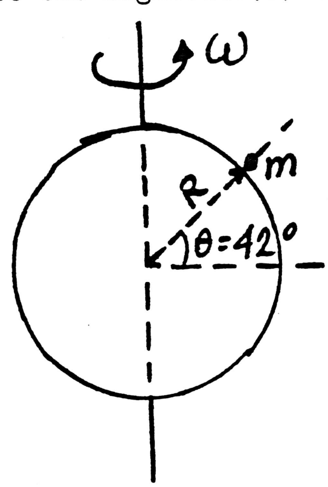
Falling Object

In 1903 Edwin Hall dropped an object a distance L = 23 meters. He was in Cambridge, Mass., latitude 42 degrees north. He found that it did not fall vertically.

1. Did the impact point deviate: (a) N or S of the vertical, (b) E or W of the vertical or, (c) both (a) and (b)?

2. Write an expression for the deviation and indicate whether it is (N or S), (E or W), or if your answer to 1 was (c) -- which of each.

3. Estimate the magnitude(s) of your answer(s).



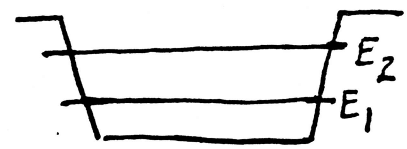
Hint: Recall that the coriolis force is given by

$$F = -2m\,(\vec{\omega} \times \vec{v})$$

II.1 (15 points)

Interacting Fermions

A certain potential well $V(\vec{x})$ supports two bound states at energy E_1 and E_2 with wave functions $\Psi_1(\vec{r}_1)$ and $\Psi_2(\vec{r}_2)$. Two spin 1/2 fermions are added to this potential well. The fermions interact via a delta function potential $U_0\delta(\vec{r}_1 - \vec{r}_2)$, where $\vec{r}_1$ and $\vec{r}_2$ denote the potitions of the two fermions and U_0 is a constant.



Write down an expression for the energy that is accurate to first order in U_0 for the lowest energy singular and triplet states.

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II.2 (5 points)

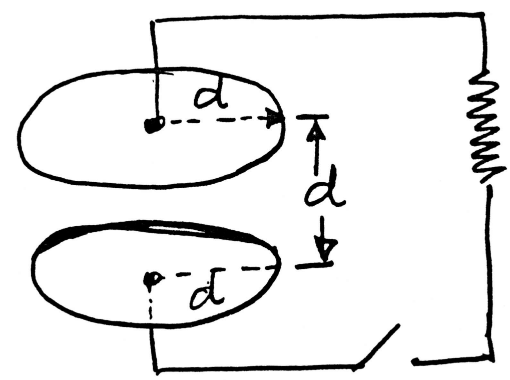
Pressure Distribution Inside the Box

N atoms of mass m form an ideal gas inside a box of LxLxL, under earth's gravity along the z direction. Derive the pressure distribution inside the box.

II.3 (5 points)

Capacitor Problem

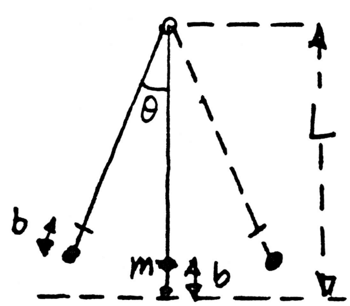
A charged capacitor initially carrying charge $\pm Q_0$ consists of two circular metal plates of radius d, separated by a distance d (see the sketch). The capacitor is discharged through a resistor, with the charge Q(t) on the top plate decreasing in a characteristic time T. The time T is much longer than d/c, where c is the speed of light. Give an order of magnitude estimate of the ratio of the total energy radiated during the discharge of the capacitor, to the electrostatic energy initially stored in the capacitor. Show that this ratio is small compared to one.



II.4 (15 points)

Swing

A child of mass m on a swing raises her center of mass by a small distance b every time the swing passes the vertical position, and lowers her mass by the same amount at each extremal position. Assuming small oscillations, calculate the work done by the child per period of oscillation. Show that the energy of the swing grows exponentially according to $dE/dt = \alpha E$ and determine the constant α.



III.1 (10 points)

Atomic Level

Consider an atom in which the last partially filled shell contains two electrons in the n = 3, $\ell = 2$ orbit in the independent particle approximation. We now allow these two electrons to interact. Ignoring spin-orbit coupling, what are the degeneracies and the ordering of the energy levels?

III.2 (15 points)

Magnet

Consider the one-dimensional magnet with Hamiltonian

$$H = -J \sum_{i=1}^{N} S_i^z S_{i+1}^z \; ; \quad \vec{S}_{N+1} \equiv \vec{S}_1, \; S = 1/2$$

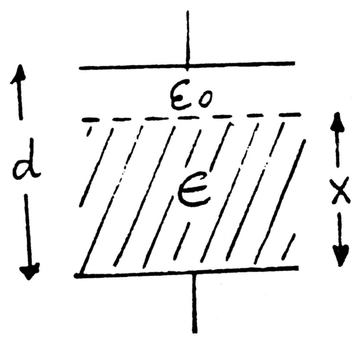
a) What are the maximum and minimum values of entropy and at which respective temperatures do they occur?

b) Consider the same model on a simple-cubic lattice. Write down the self-consistent equation of mean-field theory. Determine the critical temperature.

III.3 (5 points)

Partially filled condenser

Consider the partially filled condenser plate as shown, with an air gap



($\varepsilon_0 \simeq 1$) and a dielectric material ε.

a) Obtain an expression for the effective dielectric constant, ε'.

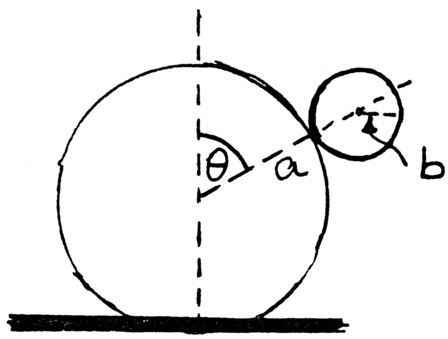
b) What is the highest effective dielectric constant possible if the air thickness is 10^{-2} of the total thickness.

III.4 (10 points)

Rolling wire hoop

A wire hoop of radius b and mass m_l per unit arc rolls without slipping over a fixed cylindrical surface of radius a, as shown in Fig. 1.

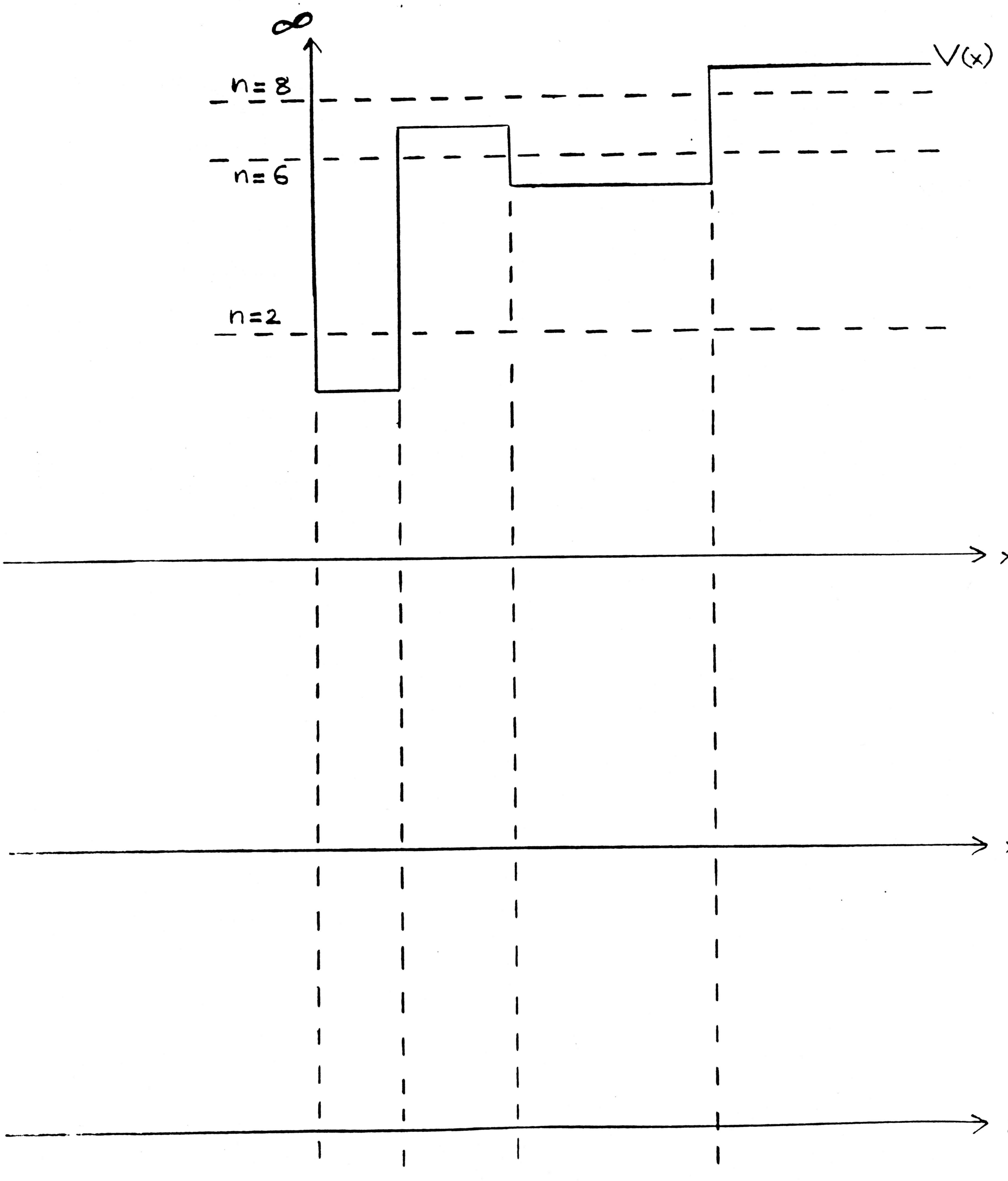
At what angle Θ will the hoop leave the cylinder?



IV.1 (15 points)

Wave function

(a) Consider the potential V(x) shown below. Sketch a possible wave function for each energy level indicated. Clearly show nodes, exponential decays, curvatures. Your sketches should illustrate the relative magnitudes of the local wavelengths and extrema in different regions. The n = 2 level is the first excited state.



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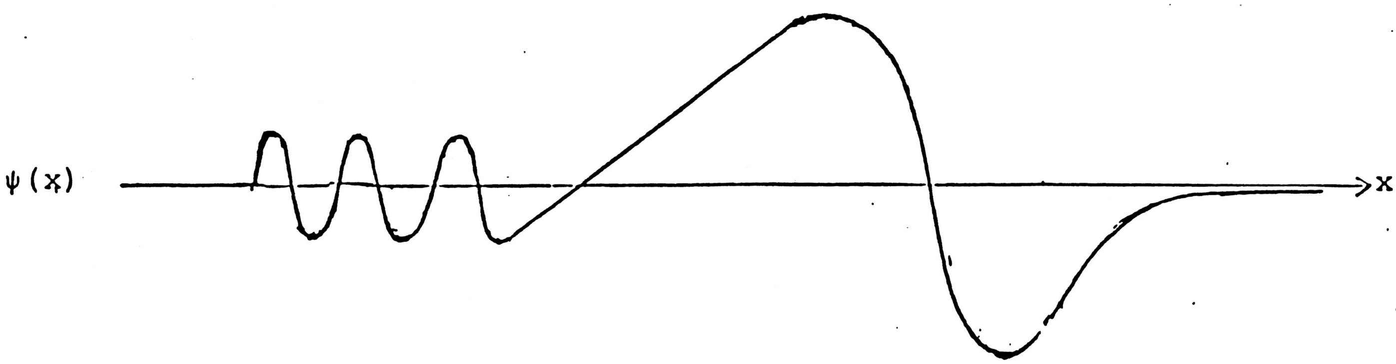
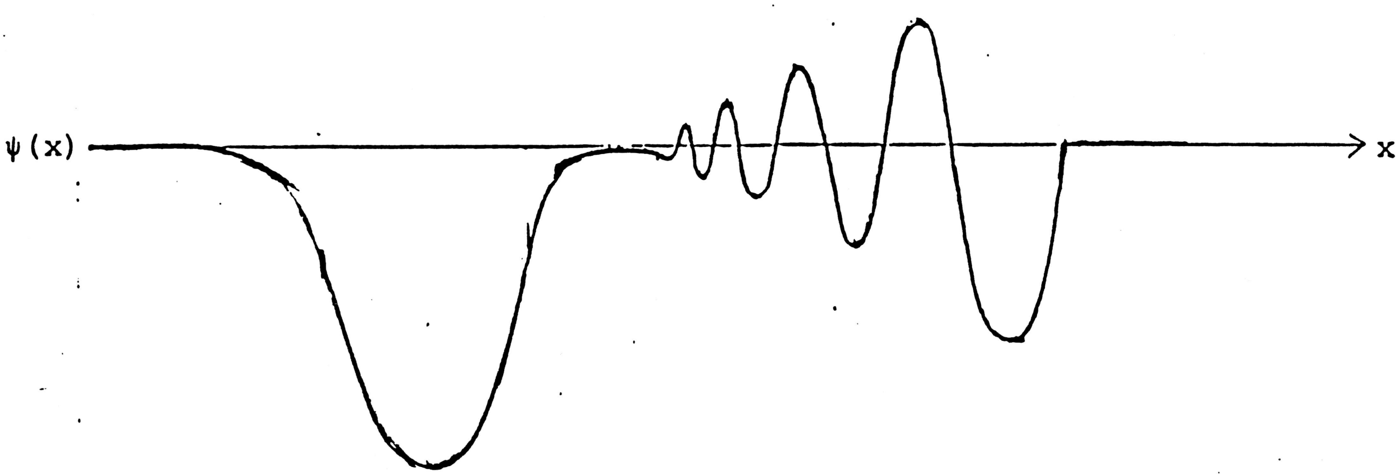
(b) Sketch a plausible potential V(x) superimposed with an energy level E_n, for each bound state wavefunction shown. Give the value of n, n = 1 being the lowest energy state.



V(x):



V(x)

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IV.2 (5 points)

Van der Waals state

a) Explain how and why the ideal gas law is modified to obtain the van der Waals equation of state.

b) Consider a gas of molecules with hard cores, as well as interactions which are either purely attractive, or purely repulsive. What can you specify in the van der Waals equation which would distinguish the two cases.

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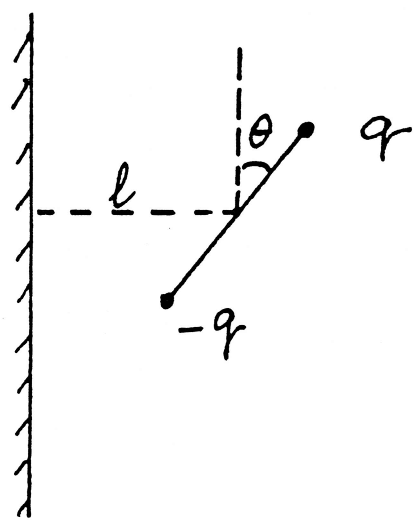
IV.3 (10 points)

Electric Dipole

An electric dipole of moment $\vec{P}$ has finite separation of the charges whose magnitudes are $|q|$. [i.e., a physical dipole] The dipole is a distance ℓ from an infinite conducting plane. Write expressions for the force, and the torque on this system.



IV.4 (10 points)

Collision of Mass-Spring System

A mass m_1, initial velocity V_o strikes a mass-spring system, m_2, initially at rest but which can recoil. The spring is massless.

1. What is the maximum compression of the spring?

2. If, long after collision, both objects travel in the same direction, what are the final velocities V_1 and V_2?

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Massachusetts Institute of Technology

Department of Physics

February 15, 1985

Doctoral General Examination

Part II

Five Hours

Instructions

This examination consists of four sections with two problems each. Solve one problem from EACH OF THE FOUR SECTIONS (a total of four problems). It is advisable to read carefully both problems in each section before making your choice. The problems in each section have been chosen to be of comparable difficulty. USE A SEPARATE EXAMINATION BOOKLET FOR EACH PROBLEM AND LABEL IT WITH THE PROBLEM NUMBER AND YOUR NAME.

NO books or references may be used.

You may find some of the information on the next pages useful.

Constant	Symbol		MKS	cgs
Boltzmann constant	k	=	$1.38\mathrm{x}10^{-23} J/^{o}K$	$1.38\mathrm{x}10^{-16} erg/^{o}K$
Electron mass	m_e	=	$9.1\mathrm{x}10^{-31}$ kg	$9.1\mathrm{x}10^{-28}$ g
Proton mass	m_p	=	$1.67\mathrm{x}10^{-27}$ kdg	$1.67\mathrm{x}10^{-24}$ g
Gravitational constant	G	=	$6.67\mathrm{x}10^{-11}$ $N\text{-}m^2/kg^2$	$6.67\mathrm{x}10^{-8}$ $dyne\text{-}cm^2/g^2$
Planck's constant	$h/2\pi$	=	$1.05\mathrm{x}10^{-34}$ J-s $h/2\pi = 6.58\mathrm{x}10^{22}$ MeV sec	$1.05\mathrm{x}10^{-27}$ erg-sec
Speed of light	c	=	$3.0\mathrm{x}10^{8}$ m/sec $hc/2\pi = 1.97\mathrm{x}10^{-11}$ MeV cm	$3.0\mathrm{x}10^{10}$ cm/sec
Electronic charge	e	=	$1.6\mathrm{x}10^{-19}$ coul	$4.8\mathrm{x}10^{-10}$ esu
Energy units	1 eV	=	$1.6\mathrm{x}10^{-19}$ J	$1/6\mathrm{x}10^{-12}$ erg
Permittivity of vacuum	ϵ_0	=	$8.85\mathrm{x}10^{-12}$ farad/m	
Permeability of vacuum	μ_0	=	$1.26\mathrm{x}10^{-6}$ henry/m	
Acceleration of gravity	g	=	9.8 m/sec^2	980 cm/sec^2
Stefan-Boltzmann constant	σ	=	$5.7\mathrm{x}10^{-8}$ $Wm^{-2}K^{-4}$	$5.7\mathrm{x}10^{-5}$ erg $s^{-1}cm^{-2}K^{-4}$
Binding energy of electron in hydrogen ground state		=	13.6 eV	
Bohr magneton	μ_B	=	$e(h/2\pi)/(2mc) = 9.27\mathrm{x}10^{-24}$ joules/tesla $= 9.27\mathrm{x}10^{-21}$ erg/gauss	
1 atmosphere of pressure		=	1.013×10^{5} $newton/m^2$	1.013×10^{6} $dynes/cm^2$
Avogadro's number		=		6.025×10^{23} /mole

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INTEGRALS

$\int \sin x \, dx = -\cos x$

$\int \cos x \, dx = \sin x$

$\int x \sin x \, dx = \sin x - x \cos x$

$\int x \cos x \, dx = \cos x + x \sin x$

$\int \sinh x \, dx = \cosh x$

$\int \cosh x \, dx = \sinh x$

DERIVATIVES

$\frac{d}{dx} \sinh x = \cosh x$

$\frac{d}{dx} \cosh x = \sinh x$

$\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$

$\frac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \tanh x$

$\frac{d}{dx} \operatorname{csch} x = -\operatorname{csch} x \coth x$

$\frac{d}{dx} \coth x = -\operatorname{csch}^2 x$

DEFINITE INTEGRALS

$$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}} \quad ; \quad \int_0^\infty dx \, x^{\frac{1}{2}} \exp(-ax) = \frac{\pi^{\frac{1}{2}}}{2a^{3/2}}$$

$$\int_{-\infty}^{\infty} x^{2n} e^{-x^2/2a^2} dx = (2\pi a^2)^{1/2} \, a^{2n} \times 1 \cdot 3 \cdot 5 \cdots (2n-1)$$

SERIES

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)x^2}{2!} \pm \cdots \text{ for } x^2 < 1$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \text{ for } |x| \leq 1$$

$$\log(N!) = N \ln N - N + \cdots\cdots$$

FORMULAS OF VECTOR ANALYSIS

	Rectangular coordinates	Cylindrical coordinates	Spherical coordinates
Conversion to rectangular coordinates		$x=r\cos\varphi \quad y=r\sin\varphi \quad z=z$	$x=r\cos\varphi\sin\theta \quad y=r\sin\varphi\sin\theta \quad z=r\cos\theta$
Gradient............	$\nabla\phi=\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}$	$\nabla\phi=\frac{\partial\phi}{\partial r}\mathbf{r}+\frac{1}{r}\frac{\partial\phi}{\partial\varphi}\boldsymbol{\phi}+\frac{\partial\phi}{\partial z}\mathbf{k}$	$\nabla\phi=\frac{\partial\phi}{\partial r}\mathbf{r}+\frac{1}{r}\frac{\partial\phi}{\partial\theta}\boldsymbol{\theta}+\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\boldsymbol{\phi}$
Divergence..........	$\nabla\cdot\mathbf{A}=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}$	$\nabla\cdot\mathbf{A}=\frac{1}{r}\frac{\partial(rA_r)}{\partial r}+\frac{1}{r}\frac{\partial A_\varphi}{\partial\varphi}+\frac{\partial A_z}{\partial z}$	$\nabla\cdot\mathbf{A}=\frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(A_\theta\sin\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$
Curl...............	$\nabla\times\mathbf{A}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ A_x&A_y&A_z\end{vmatrix}$	$\nabla\times\mathbf{A}=\begin{vmatrix}\frac{1}{r}\mathbf{r}&\boldsymbol{\phi}&\frac{1}{r}\mathbf{k}\\ \frac{\partial}{\partial r}&\frac{\partial}{\partial\varphi}&\frac{\partial}{\partial z}\\ A_r&rA_\varphi&A_z\end{vmatrix}$	$\nabla\times\mathbf{A}=\begin{vmatrix}\frac{\mathbf{r}}{r^2\sin\theta}&\frac{\boldsymbol{\theta}}{r\sin\theta}&\frac{\boldsymbol{\phi}}{r}\\ \frac{\partial}{\partial r}&\frac{\partial}{\partial\theta}&\frac{\partial}{\partial\varphi}\\ A_r&rA_\theta&rA_\varphi\sin\theta\end{vmatrix}$
Laplacian...........	$\nabla^2\phi=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$	$\nabla^2\phi=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2}+\frac{\partial^2\phi}{\partial z^2}$	$\nabla^2\phi=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\nabla\times\nabla\psi=0$$

$$\nabla\cdot(\nabla\times\mathbf{a})=0$$

$$\nabla\times(\nabla\times\mathbf{a})=\nabla(\nabla\cdot\mathbf{a})-\nabla^2\mathbf{a}$$

$$\nabla\cdot(\psi\mathbf{a})=\mathbf{a}\cdot\nabla\psi+\psi\nabla\cdot\mathbf{a}$$

$$\nabla\times(\psi\mathbf{a})=\nabla\psi\times\mathbf{a}+\psi\nabla\times\mathbf{a}$$

$$\nabla(\mathbf{a}\cdot\mathbf{b})=(\mathbf{a}\cdot\nabla)\mathbf{b}+(\mathbf{b}\cdot\nabla)\mathbf{a}+\mathbf{a}\times(\nabla\times\mathbf{b})+\mathbf{b}\times(\nabla\times\mathbf{a})$$

$$\nabla\cdot(\mathbf{a}\times\mathbf{b})=\mathbf{b}\cdot(\nabla\times\mathbf{a})-\mathbf{a}\cdot(\nabla\times\mathbf{b})$$

$$\nabla\times(\mathbf{a}\times\mathbf{b})=\mathbf{a}(\nabla\cdot\mathbf{b})-\mathbf{b}(\nabla\cdot\mathbf{a})+(\mathbf{b}\cdot\nabla)\mathbf{a}-(\mathbf{a}\cdot\nabla)\mathbf{b}$$

If $\mathbf{x}$ is the coordinate of a point with respect to some origin, with magnitude $r=|\mathbf{x}|$, and $\mathbf{n}=\mathbf{x}/r$ is a unit radial vector, then

$$\nabla\cdot\mathbf{x}=3 \qquad \nabla\times\mathbf{x}=0$$

$$\nabla\cdot\mathbf{n}=\frac{2}{r} \qquad \nabla\times\mathbf{n}=0$$

$$(\mathbf{a}\cdot\nabla)\mathbf{n}=\frac{1}{r}[\mathbf{a}-\mathbf{n}(\mathbf{a}\cdot\mathbf{n})]=\frac{\mathbf{a}_\perp}{r}$$

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CLEBSCH-GORDAN COEFFICIENTS

$j_1 = \frac{1}{2} \qquad j_2 = \frac{1}{2}$:

$$\begin{array}{cc|cccc}
 & & 1 & 1 & 0 & 1 \\
 & & 1 & 0 & 0 & -1 \\
\hline
\frac{1}{2} & \frac{1}{2} & 1 & & & \\
\frac{1}{2} & -\frac{1}{2} & & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & \\
-\frac{1}{2} & \frac{1}{2} & & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & \\
-\frac{1}{2} & -\frac{1}{2} & & & & 1
\end{array}$$

$j_1 = 1 \qquad j_2 = \frac{1}{2}$:

$$\begin{array}{cc|cccccc}
 & & \frac{3}{2} & \frac{3}{2} & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & \frac{3}{2} \\
 & & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} \\
\hline
1 & \frac{1}{2} & 1 & & & & & \\
1 & -\frac{1}{2} & & \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & & & \\
0 & \frac{1}{2} & & \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{3}} & & & \\
0 & -\frac{1}{2} & & & & \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & \\
-1 & \frac{1}{2} & & & & \sqrt{\frac{1}{3}} & -\sqrt{\frac{2}{3}} & \\
-1 & -\frac{1}{2} & & & & & & 1
\end{array}$$

$j_1 = 1 \qquad j_2 = 1$:

$$\begin{array}{cc|ccccccccc}
 & & 2 & 2 & 1 & 2 & 1 & 0 & 2 & 1 & 2 \\
 & & 2 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -2 \\
\hline
1 & 1 & 1 & & & & & & & & \\
1 & 0 & & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & & & & & & \\
0 & 1 & & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & & & & & & \\
1 & -1 & & & & \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{3}} & & & \\
0 & 0 & & & & \sqrt{\frac{2}{3}} & 0 & -\sqrt{\frac{1}{3}} & & & \\
-1 & 1 & & & & \sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{3}} & & & \\
0 & -1 & & & & & & & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & \\
-1 & 0 & & & & & & & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & \\
-1 & -1 & & & & & & & & & 1
\end{array}$$

---

d FUNCTIONS

$$d^{j}_{m'm}(\beta) \equiv \langle jm' |\exp(-i\beta J_y/\hbar)| jm \rangle$$

$j = \frac{1}{2}$:

$m' \backslash m$	$+\frac{1}{2}$	$-\frac{1}{2}$
$+\frac{1}{2}$	$\cos\beta/2$	$-\sin\beta/2$
$-\frac{1}{2}$	$\sin\beta/2$	$\cos\beta/2$

$j = 1$:

$m' \backslash m$	$+1$	0	-1
$+1$	$\frac{1}{2}(1 + \cos\beta)$	$-\frac{1}{\sqrt{2}}\sin\beta$	$\frac{1}{2}(1 - \cos\beta)$
0	$\frac{1}{\sqrt{2}}\sin\beta$	$\cos\beta$	$-\frac{1}{\sqrt{2}}\sin\beta$
-1	$\frac{1}{2}(1 - \cos\beta)$	$\frac{1}{\sqrt{2}}\sin\beta$	$\frac{1}{2}(1 + \cos\beta)$

MORE FORMULAS

$$L_\pm|\ell,m_\ell\rangle = \sqrt{\ell(\ell+1) - m_\ell(m_\ell \pm 1)}\,\hbar\,|\ell,m_\ell \pm 1\rangle$$

$$S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad S_y = \frac{\hbar}{2i}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \qquad S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$ Spin $\frac{1}{2}$ matrices

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} \qquad S_y = \frac{\hbar}{\sqrt{2}i}\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 1\\ 0 & -1 & 0\end{pmatrix} \qquad S_z = \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$$ Spin 1 matrices

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) - \frac{\hat{L}^2(\theta,\phi)\psi}{\hbar^2 r^2}$$ Spherical coordinates

$$E = -\vec{\mu}\cdot\vec{B}$$

$$\vec{\mu} = -g_L \mu_B \vec{J}/\hbar \qquad g_L = 1 + \frac{\hat{\vec{S}}\cdot\hat{\vec{J}}}{J^2}$$

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I.1 Coulomb Potential in Two Dimensions

A particle of mass m and charge $-e$ is constrained to move in the xy plane. A nucleus of charge $+Ze$ is fixed at the origin.

(a) Write the time-dependent Schroedinger equation. Effect the separation of variables, obtaining separate radial and angular equations.

(b) Solve the angular equation and specify all possible angular wavefunctions.

(c) In the radial equation, make the substitution $R(r) = r^{-a}u(r)$. Determine the value of a that reduces the radial equation to a one-dimensional Schroedinger equation. Give the effective potential of this one-dimensional Schroedinger equation.

(d) Derive the long-distance behavior of $u(r)$ for bound states.

(e) Assuming $u(0) = 0$, derive the short distance behavior of $u(r)$.

(f) For the simplest angular wavefunction, sketch $u(r)$ of the lowest three bound states, contrasting your sketches by superposing them on the same set of axes. Separately sketch the lowest bound-state wavefunction for the next-simplest angular wavefunction.

(g) Coulomb Potential in Two Dimensions in a Wedge: The following potential is added to the problem above:

$$V(\psi) \begin{cases} \nearrow 0 \text{ for } 0 < \phi < \phi_o \\ \searrow \infty \text{ for } \phi_o < \phi < 2\pi \, . \end{cases}$$

Which of the answers to parts (a) through (f) change, and how?

I.2 Quantum Spin Chain

Consider a one-dimensional chain of N spins of S = 1/2 coupled by the Hamiltonian

$$H = J \sum_{i=1}^{N-1} \vec{S}_i \cdot \vec{S}_{i+1}$$

where $\vec{S} = (S_x, S_y, S_z)$ are the spin operators and $J > 0$. The ground state energy per spin of this Hamiltonian was found to be $E_o = - J(\ln 2 - 1/4) = -0.433J$ in a famous paper by H. Bethe in 1931. For this examination, you will not be required to reproduce this solution. Instead, we will determine upper and lower bounds for the ground state energy.

(a) If the spin operators are treated as c-numbers, what is the ground state spin configuration and what is the ground state energy?

(b) Consider the following trial wavefunction

$$\Phi = \Pi_{i \text{ odd}} \phi_i$$

where ϕ_i is the singlet state formed from the spin on sites i and i + 1. Calculate the variational energy corresponding to this trial wavefunction and compare it with the classical energy. What is an upper bound to the ground state energy?

(c) Prove the following lower bound

$$-\frac{3}{4} J < E_o$$

Hint: First prove the following:

$$\langle\alpha| \vec{S}_i \cdot \vec{S}_j |\alpha\rangle > E_{min}$$

for any state $|\alpha\rangle$

if $\vec{S}_i$, $\vec{S}_j$ are spin S operators, and E_{min} is the smallest eigenvalue of $\vec{S}_i \cdot \vec{S}_j$.

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II.1 Interacting Particles

Consider two particles each of mass m, moving in one-dimensional space. They are subject to an external harmonic potential, and they also interact harmonically, i.e.,

$$V(x_1, x_2) = \frac{1}{2} K (x_1^2 + x_2^2) + \frac{1}{2} K' (x_1 - x_2)^2$$

where x_1, x_2 denote the position of the particles.

What is the mean square distance bewteen the particles $<(x_1 - x_2)^2>$ if the system is in thermal equilibrium at a temperature T?

Consider two different cases:

1. The particles are classical.

2. The particles are quantum mechanical bosons.

II.2

Equation of State of Quantum Gas

Consider a non-interacting, one-component quantum gas at temperature T, with chemical potential μ, in a cubic volume V. Treat the separate cases of bosons and fermions.

a) For a dilute such system, derive the equation of state in terms of temperature T, pressure P, density ρ, mass m, Planck constant h, and Boltzmann constant k. Do this derivation approximately by keeping the leading and next-leading powers of ρ. Use the relation

$$PV = kT \,\ell n\, Z,$$

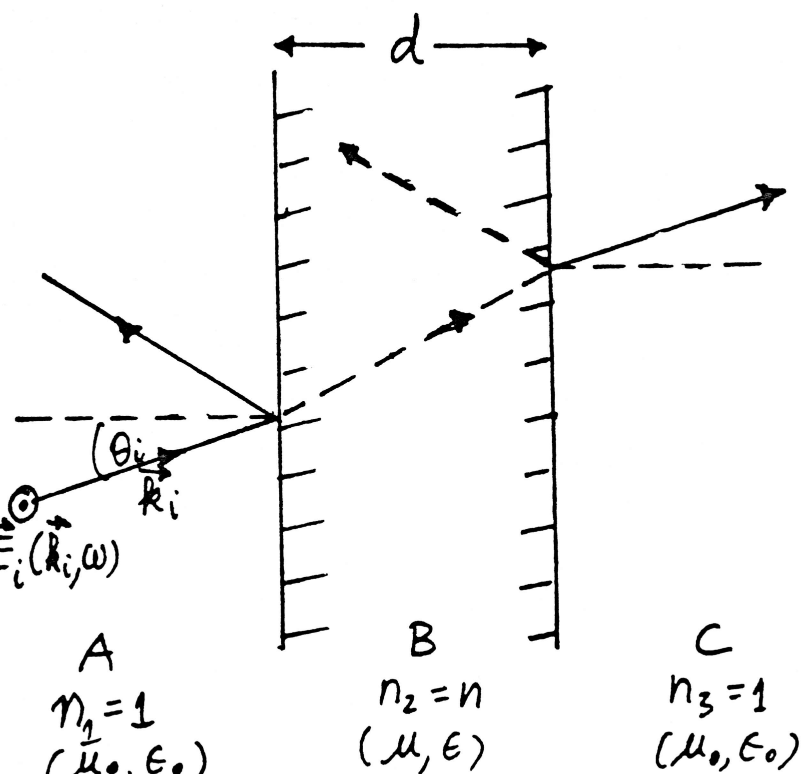
where V is volume, and Z is the grand-canonical partition function.

b) Interpret your results to a) in terms of effective classical systems.

c) At a given temperature, for which densities are your results to a) valid?

III.1 Transmission of EM Wave through a Dielectric Slab

Consider the dielectric slab of thickness d in empty space, with electromagnetic radiation with frequency ω, and propagation vector $\vec{k}_i$, incident on it in region A, as shown in the figure. The angle of incidence is θ_i, and the polarization is such that $\vec{E}_i$, the incident electric field points out of the page as shown. The index of refraction in the dielectric is n, and it may be a function of frequency, ω.



(a) Obtain the transmission coefficient from region A to region C as a function of θ_i after the first transit through the slab.

(b) Now consider the case of a dilute plasma where $n^2 = 1 - \omega_p^2/\omega^2$, where ω_p is the electron plasma frequency ($\omega_p^2 = 4\pi Ne^2/m_e$). Discuss the behavior of the transmission coefficient for arbitrary values of ω_p/ω as θ_i is varied in the range $0 < \theta_i < \pi/2$. In particular, find the critical angle θ_{ic} (ω_p/ω) for which no transmission occurs.

III.1 -- Continued

(c) Prove that for $\theta > \theta_{ic}$ complete reflection occurs if d is large. (Hint: show that in the limit $d \to \infty$, $\vec{S} \cdot \hat{n} = 0$ in region B, where $\vec{S}$ is the Poynting vector and $\hat{n}$ is the normal to the interface between regions A and B.)

(d) Reducing d, the width of the plasma layer, show that significant transmission of wave power can be achieved even for $\theta > \theta_{ic}$ if d is small enough. Obtain an approximate condition on d for such power transmission. Explain the physics.

III.2 Plasma Waves

A transverse ($\vec{\nabla} \cdot \vec{E} = 0$) electromagnetic wave propagates in a dilute plasma with n_o electrons per cubic centimeter. There is no static magnetic field in the plasma. For our purposes, we can take the positive ions in the plasma to be infinitely massive, with the equation of motion of the electrons (frictionally coupled to the positive ions) given by

$$m_e \frac{d\vec{v}_e}{dt} = - e \left(\vec{E} + \frac{1}{c} \vec{v}_e \times \vec{B}\right) - m_e \nu_e \vec{v}_e \qquad (1)$$

where m_e is the electron mass, $-e$ is the electron charge, and ν_e is a collision frequency.

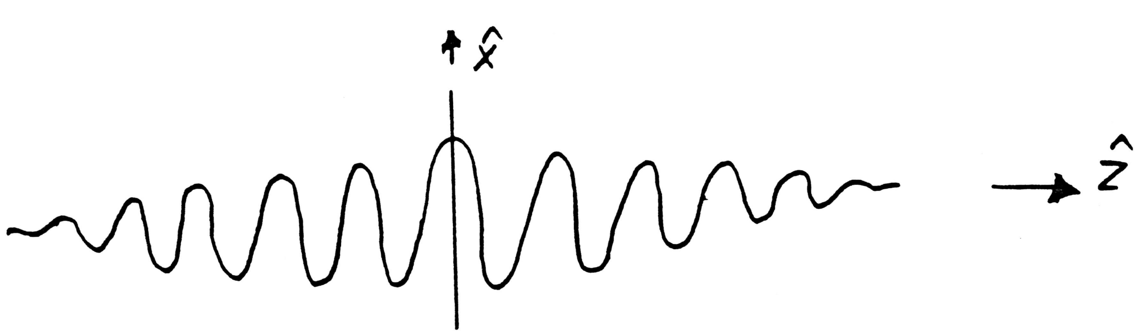
(a) Show that the "generalized Ohm's law" for this situation is

$$\frac{\partial \vec{J}}{\partial t} + \nu_e \vec{J} = \frac{\omega_p^2}{4\pi} \vec{E} \qquad (2)$$

where $\omega_p/2\pi$ is the plasma frequency, and $\omega_p^2 = 4\pi n_o e^2/m_e$.

(b) With the results of (a), derive a dispersion relation for transverse ($\vec{\nabla} \cdot \vec{E} = 0$) electromagnetic waves in this plasma.

(c) A planar antenna sits in this plasma at $z = 0$, carrying externally driven currents oscillating at frequency $\omega_o >> \omega_p, \nu_e$. The antenna generates electromagnetic plane waves propagating into $z > 0$ and $z < 0$, polarized in the $\hat{x}$ direction (see sketch).



III.2 -- Continued

Using your disperion relation from (b), show that if we keep only the leading term in both the real and imaginary parts of k, then the electric field in the plasma varies as

$$\vec{E}(z,t) = \hat{x}E_o e^{-|z|/L_o} \cos\omega_o \left(\frac{|z|}{c} - t\right) . \qquad (3)$$

What is L_o in terms of ω_p, ν_e, ω_o, and the speed of light c?

(d) The wave propagating into $z > 0$ is carrying electromagnetic energy into $z > 0$. The electromagnetic energy flux is less and less as z becomes larger and larger. Give a qualitative reason for why this happens. Then show quantitatively and explicitly using your solutions above that your qualitative reason is correct, keeping terms only to the same order as used in deriving Eq. (3). It is easiest to deal with time-averaged quantities only (e.g., the electromagnetic energy flux time averaged over one period $2\pi/\omega_o$).

(e) The wave propagating into $z > 0$ is also carrying electromagnetic momentum into $z > 0$. This electromagnetic momentum flux is less and less as z becomes larger and larger. As above, give a qualitative reason why this happens, and then show quantitatively that your qualitative reason is correct.

Hints to (d) and (e):

Energy conservation (Poynting's Theorem):

$$\frac{\partial}{\partial t}\frac{(E^2 + B^2)}{8\pi} + \nabla \cdot \left(\frac{c}{4\pi}\vec{E} \times \vec{B}\right) = -\vec{J} \cdot \vec{E}$$

Momentum conservation:

$$\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\vec{E} \times \vec{B}}{4\pi}\right) + \nabla \cdot [-\overleftrightarrow{T}] = -\frac{1}{c}\vec{J} \times \vec{B}$$

where $\overleftrightarrow{T} = (1/4\pi)\,[\vec{E}\vec{E} + \vec{B}\vec{B} - (1/2)\,\overleftrightarrow{I}\,(\vec{E}^2 + \vec{B}^2)]$.

IV.1 Spinning Coin

A coin of radius "a" initially spins on a smooth (frictionless) table about a vertical axis coincident with a diameter. Assume that the coin is an infinitely thin disk. We wish to study the stability of its motion.

(a) Determine the moments of inertia.

(b) Find the Lagrangian for arbitrary angles and then determine the integrals of motion.

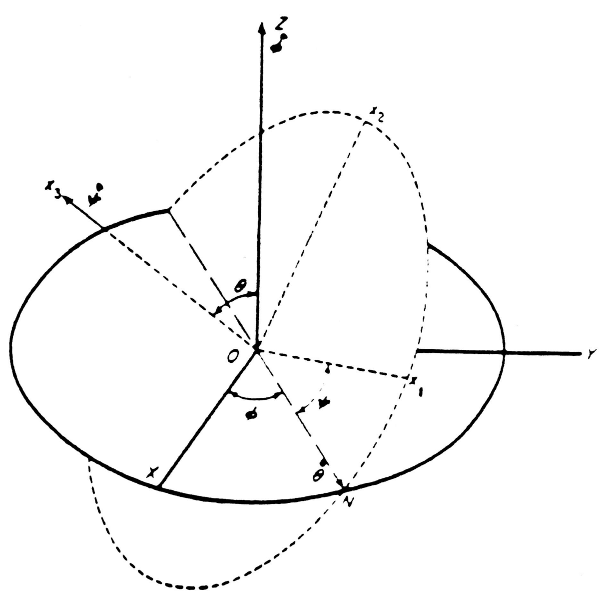
(Hint: use Eulerian angles.)

(c) Using the results of (b), obtain the equation of motion in terms of ω, the initial angular velocity of the spinning coin.

(d) Show that the motion is stable if $\omega > 2\sqrt{g/a}$.

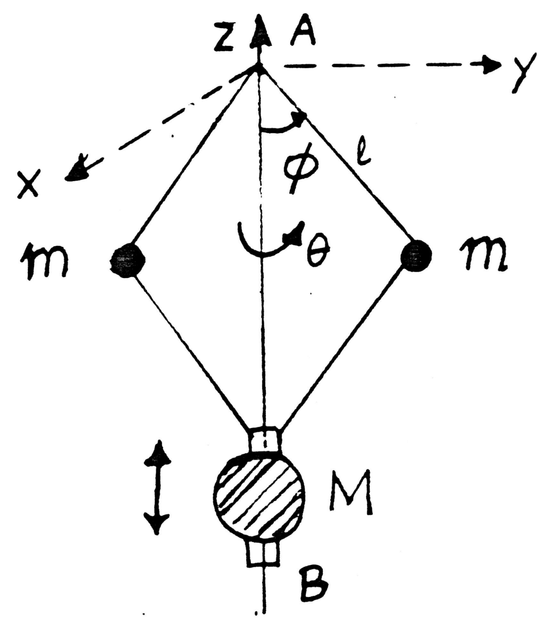
(Hint: to prove (d), note that the equation of motion, $\dot{\theta}^2 = f(\theta)$ should have double roots at $\theta = \pi/2$, and that $d^2 f(\theta)/d\theta^2$ be negative at $\theta = \pi/2$.)

Eulerian angles.



IV.2 "Governor"

Consider the motion of the governor shown in the figure. The massless rigid rods all length ℓ, are pivoted at A and at each mass, in such a way that, as the masses m move outward, the mass M can move smoothly in the vertical direction along the rod A-B in the z direction. The system rotates about A-B so that at any instant its position is given by the angle θ relative to the x-axis. The corresponding angular velocity of the rotation is $\dot{\theta}$. Let ϕ denote that angle which one of the bars makes with A-B. $\dot{\phi}$ is the corresponding angular velocity. Finally, let I denote the moment of intertia of M about the A-B axis.



(a) Find the Lagrangian of the system in terms of generalized coordinates.

(b) Find Lagrange's equations. Is the angular momentum conserved?

(c) Assume that initially the system is at rest ($\phi(t = 0) = 0$). Then start up the system by giving it an angular velocity $\dot{\theta}$. Discuss qualitatively the behavior of $\dot{\theta}$ for arbitrary values of ϕ as time increases.

(d) Obtain an expression for the maximum position, ($\dot{\phi} = \ddot{\phi} = 0$) of the governor for arbitrary values of ϕ. At this position deduce an expression for the angular momentum in terms of I, m, M, ℓ, g, and ϕ. Thus, obtain an expression for $\omega = \dot{\theta}(t = 0)$ if we wish $\phi = 45°$ at the maximum point.

IV.2 -- Continued

(e) Consider the motion for small values of ϕ($\phi \ll 1$) by expanding the transcendential functions in powers of ϕ. Then, keeping only terms linear in ϕ, $\dot{\phi}$, $\ddot{\phi}$, solve the equations of motion exactly. Assume that at $t = 0$, $\phi = 0$, and obtain $z(t)$, the position of the mass M.

(f) Describe the motion of the governor for $\zeta^2 < 1$, and $\zeta^2 > 1$, where $\zeta^2 = \dot{\theta}^2 \ell/g(1 + M/m)$. Give a physical interpretation.

MIT

Massachusetts Institute of Technology

Department of Physics

September 24, 1985

9:30 am - 2:30 pm

Doctoral General Examination

Part I

Five Hours

Instructions

This examination is divided into four groups of questions. USE A SEPARATE EXAMINATION BOOKLET FOR EACH GROUP. Be sure to list the problem number with each solution.

A diagram or sketch as part of the answer is often useful, particularly when problem calls for a qualitative response.

NO books or reference materials may be used. You may find some of the information on the next pages of help in your work.

Constant	Symbol		MKS	cgs
Boltzmann constant	k	=	$1.38 \times 10^{-23} J/^{o}K$	$1.38 \times 10^{-16} erg/^{o}K$
Electron mass	m_e	=	9.1×10^{-31} kg	9.1×10^{-28} g
Proton mass	m_p	=	1.67×10^{-27} kdg	1.67×10^{-24} g
Gravitational constant	G	=	6.67×10^{-11} $N\text{-}m^2/kg^2$	6.67×10^{-8} $dyne\text{-}cm^2/g^2$
Planck's constant	$h/2\pi$	=	1.05×10^{-34} J-s $h/2\pi = 6.58 \times 10^{22}$ MeV sec	1.05×10^{-27} erg-sec
Speed of light	c	=	3.0×10^{8} m/sec $hc/2\pi = 1.97 \times 10^{-11}$ MeV cm	3.0×10^{10} cm/sec
Electronic charge	e	=	1.6×10^{-19} coul	4.8×10^{-10} esu
Energy units	1 eV	=	1.6×10^{-19} J	$1/6 \times 10^{-12}$ erg
Permittivity of vacuum	ϵ_0	=	8.85×10^{-12} farad/m	
Permeability of vacuum	μ_0	=	1.26×10^{-6} henry/m	
Acceleration of gravity	g	=	$9.8\ m/sec^2$	$980\ cm/sec^2$
Stefan-Boltzmann constant	σ	=	$5.7 \times 10^{-8}\ Wm^{-2}K^{-4}$	5.7×10^{-5} erg $s^{-1}cm^{-2}K^{-4}$
Binding energy of electron in hydrogen ground state		=	13.6 eV	
Bohr magneton	μ_B	=	$e(h/2\pi)/(2mc) = 9.27 \times 10^{-24}$ joules/tesla $= 9.27 \times 10^{-21}$ erg/gauss	
1 atmosphere of pressure		=	$1.013 \times 10^{5}\ newton/m^2$	$1.013 \times 10^{6}\ dynes/cm^2$
Avogadro's number		=		6.025×10^{23} /mole

MHC

INTEGRALS

$\int \sin x \, dx = -\cos x$

$\int \cos x \, dx = \sin x$

$\int x \sin x \, dx = \sin x - x \cos x$

$\int x \cos x \, dx = \cos x + x \sin x$

$\int \sinh x \, dx = \cosh x$

$\int \cosh x \, dx = \sinh x$

DERIVATIVES

$\frac{d}{dx} \sinh x = \cosh x$

$\frac{d}{dx} \cosh x = \sinh x$

$\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$

$\frac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \tanh x$

$\frac{d}{dx} \operatorname{csch} x = -\operatorname{csch} x \coth x$

$\frac{d}{dx} \coth x = -\operatorname{csch}^2 x$

DEFINITE INTEGRALS

$$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

$$\int_{-\infty}^{\infty} x^{2n} e^{-x^2/2a^2} dx = (2\pi a^2)^{1/2} a^{2n} \times 1\cdot3\cdot5\cdots(2n-1)$$

SERIES

$$(1\pm x)^n = 1 \pm nx + \frac{n(n-1)x^2}{2!} \pm \cdots \text{ for } x^2 < 1$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \text{ for } |x| \leq 1$$

$$\log(N!) = N \ln N - N + \cdots\cdots$$

FORMULAS OF VECTOR ANALYSIS

	Rectangular coordinates	Cylindrical coordinates	Spherical coordinates
Conversion to rectangular coordinates		$x = r\cos\varphi \quad y = r\sin\varphi \quad z = z$	$x = r\cos\varphi\sin\theta \quad y = r\sin\varphi\sin\theta \quad z = r\cos\theta$
Gradient	$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$	$\nabla\phi = \frac{\partial\phi}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial\phi}{\partial\varphi}\boldsymbol{\phi} + \frac{\partial\phi}{\partial z}\mathbf{k}$	$\nabla\phi = \frac{\partial\phi}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\boldsymbol{\theta} + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\boldsymbol{\phi}$
Divergence	$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\nabla\cdot\mathbf{A} = \frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}$	$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(A_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$
Curl	$\nabla\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$	$\nabla\times\mathbf{A} = \begin{vmatrix} \frac{1}{r}\mathbf{r} & \boldsymbol{\phi} & \frac{1}{r}\mathbf{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ A_r & rA_\varphi & A_z \end{vmatrix}$	$\nabla\times\mathbf{A} = \begin{vmatrix} \frac{\mathbf{r}}{r^2\sin\theta} & \frac{\boldsymbol{\theta}}{r\sin\theta} & \frac{\boldsymbol{\phi}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ A_r & rA_\theta & rA_\varphi\sin\theta \end{vmatrix}$
Laplacian	$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$	$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2}$	$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$
$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$
$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$
$$\nabla\times\nabla\psi = 0$$
$$\nabla\cdot(\nabla\times\mathbf{a}) = 0$$
$$\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$$
$$\nabla\cdot(\psi\mathbf{a}) = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$$
$$\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a}$$
$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$
$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$$
$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}$$

If $\mathbf{x}$ is the coordinate of a point with respect to some origin, with magnitude $r = |\mathbf{x}|$, and $\mathbf{n} = \mathbf{x}/r$ is a unit radial vector, then

$$\nabla\cdot\mathbf{x} = 3 \qquad \nabla\times\mathbf{x} = 0$$
$$\nabla\cdot\mathbf{n} = \frac{2}{r} \qquad \nabla\times\mathbf{n} = 0$$
$$(\mathbf{a}\cdot\nabla)\mathbf{n} = \frac{1}{r}[\mathbf{a} - \mathbf{n}(\mathbf{a}\cdot\mathbf{n})] = \frac{\mathbf{a}_\perp}{r}$$

CLEBSCH-GORDAN COEFFICIENTS

$j_1 = \frac{1}{2} \qquad j_2 = \frac{1}{2}$:

m_1	m_2	1, 1	1, 0	0, 0	1, −1
$\frac{1}{2}$	$\frac{1}{2}$	1			
$\frac{1}{2}$	$-\frac{1}{2}$		$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	
$-\frac{1}{2}$	$\frac{1}{2}$		$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$	
$-\frac{1}{2}$	$-\frac{1}{2}$				1

$j_1 = 1 \qquad j_2 = \frac{1}{2}$:

m_1	m_2	$\frac{3}{2}, \frac{3}{2}$	$\frac{3}{2}, \frac{1}{2}$	$\frac{1}{2}, \frac{1}{2}$	$\frac{3}{2}, -\frac{1}{2}$	$\frac{1}{2}, -\frac{1}{2}$	$\frac{3}{2}, -\frac{3}{2}$
1	$\frac{1}{2}$	1					
1	$-\frac{1}{2}$		$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{3}}$			
0	$\frac{1}{2}$		$\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{1}{3}}$			
0	$-\frac{1}{2}$				$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{3}}$	
−1	$\frac{1}{2}$				$\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{2}{3}}$	
−1	$-\frac{1}{2}$						1

$j_1 = 1 \qquad j_2 = 1$:

m_1	m_2	2, 2	2, 1	1, 1	2, 0	1, 0	0, 0	2, −1	1, −1	2, −2
1	1	1								
1	0		$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$						
0	1		$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$						
1	−1				$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{3}}$			
0	0				$\sqrt{\frac{2}{3}}$	0	$-\sqrt{\frac{1}{3}}$			
−1	1				$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{3}}$			
0	−1							$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	
−1	0							$\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$	
−1	−1									1

d FUNCTIONS

$$d^{j}_{m'm}(\beta) \equiv \langle jm' | \exp(-i\beta J_y/\hbar) | jm \rangle$$

$j = \frac{1}{2}$:

$m' \backslash m$	$+\frac{1}{2}$	$-\frac{1}{2}$
$+\frac{1}{2}$	$\cos \beta/2$	$-\sin \beta/2$
$-\frac{1}{2}$	$\sin \beta/2$	$\cos \beta/2$

$j = 1$:

$m' \backslash m$	+1	0	−1
+1	$\frac{1}{2}(1 + \cos \beta)$	$-\frac{1}{\sqrt{2}} \sin \beta$	$\frac{1}{2}(1 - \cos \beta)$
0	$\frac{1}{\sqrt{2}} \sin \beta$	$\cos \beta$	$-\frac{1}{\sqrt{2}} \sin \beta$
−1	$\frac{1}{2}(1 - \cos \beta)$	$\frac{1}{\sqrt{2}} \sin \beta$	$\frac{1}{2}(1 + \cos \beta)$

MORE FORMULAS

Normalized wave functions for hydrogen: U_{nlm}

$$U100 = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_o}\right)^{3/2} e^{-Zr/a_o}$$

$$U200 = \frac{1}{\sqrt{32\pi}}\left(\frac{Z}{a_o}\right)^{3/2}\left(2 - \frac{Zr}{a_o}\right)e^{-Zr/2a_o}$$

$$U210 = \frac{1}{\sqrt{32\pi}}\left(\frac{Z}{a_o}\right)^{3/2}\frac{Zr}{a_o}\, e^{-Zr/2a_o} \cos\theta$$

$$U21\pm 1 = \frac{1}{\sqrt{64\pi}}\left(\frac{Z}{a_o}\right)^{3/2}\frac{Zr}{a_o}\, e^{-Zr/2a_o} \sin\theta\, e^{\pm i\phi}$$

Where $a_o = \dfrac{4\pi\varepsilon_o\hbar^2}{me^2}\,(1 + m/M)$

$$L_\pm|\ell,m_\ell\rangle = \sqrt{\ell(\ell+1) - m_\ell\,(m_\ell \pm 1)}\,\hbar\,|\ell,m_\ell \pm 1\rangle$$

$S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ $S_y = \frac{\hbar}{2i}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ $S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ Spin $\frac{1}{2}$ matrices

$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$ $S_y = \frac{\hbar}{\sqrt{2}i}\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 1\\ 0 & -1 & 0\end{pmatrix}$ $S_z = \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$ Spin 1 matrices

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) - \frac{\hat{L}^2(\theta,\phi)\psi}{\hbar^2 r^2}$$ Spherical coordinates

$$E = -\vec{\mu}\cdot\vec{B}$$

$$\vec{\mu} = -g_L\mu_B\,\vec{J}/\hbar \qquad g_L = 1 + \frac{\vec{S}\cdot\vec{J}}{J^2}$$

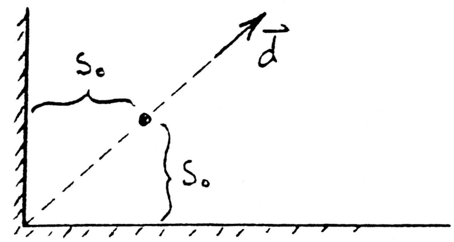
M I (

I.1 Electrostatics

An electron is at a distance S_o from each of two semi-infinite, grounded conducting sheets meeting at a right angle.

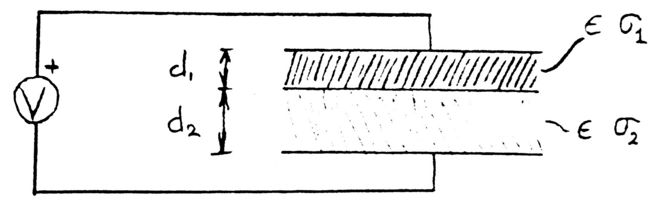
a) What is the force $\vec{F}$ exerted on the electron?

b) How much work is required to move the electron to ∞ along direction $\vec{d}$?



I.2 Capacitor

A capacitor, plate area A, employs a two-component dielectric which is also conductive, as shown. The dielectric constants ε are the same, but the conductivities $\sigma_1 \neq \sigma_2$.



With a voltage V applied:

a) What are electric fields in 1 & 2?

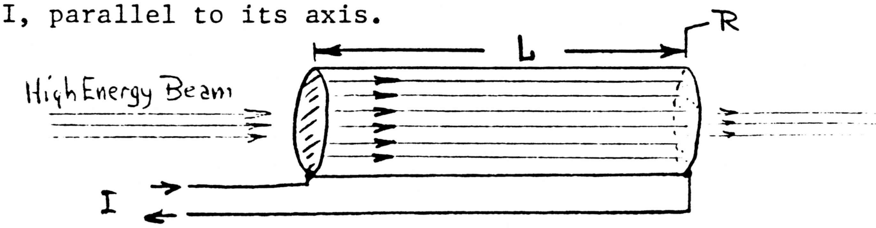
b) What is total current?

c) What is total surface charge density at interface?

d) What is the electric polarization in each dielectric?

I.3 Magnetic Lens

An unusual kind of magnetic lens for high energy charged particle beams can be made by passing a large current through a conducting plasma confined to a (finite) cylindrical region. The cylinder, length L, radius R, carries a uniform current I, parallel to its axis.



a) Find direction and magnitude of the magnetic field everywhere inside cylinder. (Ignore end effects and contribution of high energy beam current.)

b) Regard this system as a thin lens, where L << f: the focal length. A finite cross-section beam of penetrating particles (each of momentum p, positive charge) passes through cylinder from left, centered on axis of cylinder. Show that transmitted beam is focussed to a point and compute the focal length f.

c) Describe the behavior of system when the current is reversed.

I.4 Oil Drop

Consider a spherical oil drop of radius R_o, carrying a total charge Q spread uniformly over its surface. If Q exceeds a certain value the drop will break up into two. This instability can be analyzed using energy considerations. τ is the surface tension coefficient of the droplet. The total energy of the single droplet is

$$U = Q^2/(8\pi\varepsilon_o R_o) + 4\pi\tau R_o^2$$

a) Write down an equation for the energy of the separate system on the point of breakup: i.e. with the centers of the two drops a distance $2R_1$, apart. Assume for simplicity a symmetrical breakup into two equal size droplets of radius R_1, and charge $Q_1 = Q/2$. Surface charge densities are assumed uniform.

b) Find the condition for instability of the initial drop.

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II.1 Relativity

Light of frequency ν_o is reflected in the laboratory at normal incidence from a mirror moving in the opposite direction to the incident light, with velocity $V \lesssim c$.

a) What is the frequency ν of reflected light in terms of ν_o, c, and V?

b) What is energy of each reflected photon?

c) If the average incident energy flux is P_o Watts/m^2, what is average reflected energy flux in the laboratory?

II.2 Gravity Subway

A straight, evacuated tunnel connects cities A and B, a distance (measured along surface) S apart. A subway train travels by gravity alone from A → B in this tunnel.

a) Find the maximum velocity enroute.

b) How does the time of travel depend on S and on the earth's radius? (Show proof)

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II.3 Collision

Suppose that a body of mass m, frontal area A, moving with velocity V_o is brought to a halt in collision with an absorbing medium such as snow or sand.

a) Show that the average pressure P acting on the body during impact is $P = mV_o^2/(2\ Ah)$, where h is distance body sinks into medium.

b) Suppose a person ($A = 0.2\ m^2$, mass = 70 Kg) jumps out a window into a net 10 m below. How deeply should the net sink to avoid injury? $P\ max = 2 \times 10^5$ newtons/m^2.

II.4 P.P Scattering

In experiments on proton-proton scattering, where the target particle is initially at rest and the incident particle is relativistic the paths of the two recoil particles are at angle θ to one another. Find an expression for θ in terms of the initial energies and momenta of the particles.

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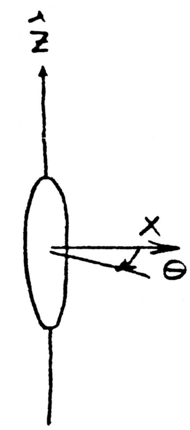
III.1 Galvanometer



Consider a mirror attached to a long massless wire as shown below. When the wire is experiencing no twisting torque the normal to the plane of the mirror points in the X direction.

If the plane of the mirror makes an angle θ with χ, the suspension wire exerts a restoring torque proportional to the deflection angle θ i.e. $\tau = \alpha\,\theta$. Let the moment of inertia of the mirror and wire be I.

a) Write down an expression for the total kinetic and potential energy of the system in terms of $\dot{\theta}$, θ I and α.

b) Using θ as the generalized coordinate of this problem obtain the generalized momentum P_θ.

c) Write down the Hamiltonian H for this system in terms of I, θ, P_θ and α.

d) Use the equipartition theorem to calculate the mean square amplitude of fluctuation in θ if the system of mirror and wire is in thermal equilibrium at temperature T.

III.2 Electron Gas

When matter is compressed to very high density, the interaction of the electrons with the nuclei and with one another becomes unimportant, and the electrons can be treated as an extremely relativistic free Fermion gas. Beyond a certain electron density n, it becomes energetically favorable for a nucleus with charge Z to absorb an electron and convert to a nucleus with the same atomic weight and charge Z-1. The energy of a nucleus of charge Z is Δ MeV lower than nucleus Z-1. Taking a typical value of Δ, for example 10 MeV, give an explicit estimate of the density n.

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III.3 Zipper

The unwinding of a two-stranded molecule is modeled by a finite zipper with N links. The energy of each link is $-\varepsilon$ and zero when it is closed and open, respectively. However, the zipper can only unzip from the right end, and a link can be open only if all links to its right are already open.

a) What is the average number of closed links at a temperature T?

b) Determine the low temperature limit.

Help: $$\sum_{n=0}^{N} X^n = \frac{1 - X^{N+1}}{1 - X}$$

III.4 Plasma

Consider an N component plasma. Each component consists of ions with charge z_i e and a density n_i, $i = 1 \cdots N$. The total charge of the plasma is zero. The temperature T is high so that we can treat the problem classically. An infinitessimal test charge q is inserted at $\vec{r} = o$.

a) Show that the electrostatic potential ϕ around the test charge satisfies the equation

$\nabla^2 \phi - K^2 \phi = -4 \pi q \delta(\vec{r})$ Under the assumption that $q\phi << KT$

b) Derive an expression for K^2.

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IV.1 Finite Proton

The charge distribution of a proton is not a point but is distributed in space with density $\rho(\vec{r})$. Find the first order shift in energy of a hydrogen atom in its ground state arising from the distributed charge in its "nucleus": ρ uniform within a sphere, radius r_n. Take advantage of the fact that $r_n \ll a_o$ in finding your answer.

IV.2 K-capture

A nucleus of charge Z has its atomic number changed to Z - 1 by nuclear K-capture beta decay. What is the probability that the K-shell electron that is not absorbed is still a K-shell electron around the new nucleus after decay. Ignore all e-e interactions. Find numerical value for $Z = 28$, the $Co^{60} \rightarrow Ni^{61}$ transition.

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IV.3 Scattering

Hard core scattering: consider an infinite hard core potential for a particle of mass M:

$$V(r) = \begin{cases} \infty & \underline{r<\alpha} \\ o & r>\alpha \end{cases}$$

One scattering solution to V(r) can be written (in the region $r > \alpha$)

$$\psi(r) = \frac{\text{Sin }(kr + \delta)}{r}$$

a) What is the energy of this solution? Evaluate the constant δ.

b) If V(r) were not infinitely repulsive but only went to a finite constant, i.e.

$$V(r) = \begin{cases} V_o & r<\alpha\,(V_0 > o) \\ 0 & r>\alpha \end{cases}$$

Would the absolute value of δ be larger or smaller than in part (a)? Why?

IV.4 Molecular Vibrations

It takes about 0.53 eV to excite the H_2 molecule from its ground to its first excited vibrational state.

a) What is the excitation energy for diatomic deuterium?

b) The H_2^+ ion has a much smaller binding energy than H_2. Is H_2^+'s vibrational excitation energy larger or smaller than H_2? Why?

c) Electron diffraction experiments show that the average internuclear separation of diatomic molecules increases as the temperature increases. Explain why?

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Massachusetts Institute of Technology

Department of Physics

September 27, 1985

9:30 am - 2:30 pm

Doctoral General Examination

Part II

Five Hours

Instructions

This examination consists of four sections with two problems each. Solve one problem from EACH OF THE FOUR SECTIONS (a total of four problems). It is advisable to read carefully both problems in each section before making your choice. The problems in each section have been chosen to be of comparable difficulty. USE A SEPARATE EXAMINATION BOOKLET FOR EACH PROBLEM AND LABEL IT WITH THE PROBLEM NUMBER AND YOUR NAME.

NO books or references may be used.

You may find some of the information on the next pages useful.

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Constant	Symbol		MKS	cgs
Boltzmann constant	k	=	$1.38 \times 10^{-23} J/^{o}K$	$1.38 \times 10^{-16} erg/^{o}K$
Electron mass	m_e	=	9.1×10^{-31} kg	9.1×10^{-28} g
Proton mass	m_p	=	1.67×10^{-27} kdg	1.67×10^{-24} g
Gravitational constant	G	=	6.67×10^{-11} $N\text{-}m^2/kg^2$	6.67×10^{-8} $dyne\text{-}cm^2/g^2$
Planck's constant	$h/2\pi$	=	1.05×10^{-34} J-s $h/2\pi = 6.58 \times 10^{22}$ MeV sec	1.05×10^{-27} erg-sec
Speed of light	c	=	3.0×10^{8} m/sec $hc/2\pi = 1.97 \times 10^{-11}$ MeV cm	3.0×10^{10} cm/sec
Electronic charge	e	=	1.6×10^{-19} coul	4.8×10^{-10} esu
Energy units	1 eV	=	1.6×10^{-19} J	$1/6 \times 10^{-12}$ erg
Permittivity of vacuum	ϵ_0	=	8.85×10^{-12} farad/m	
Permeability of vacuum	μ_0	=	1.26×10^{-6} henry/m	
Acceleration of gravity	g	=	$9.8\ m/sec^2$	$980\ cm/sec^2$
Stefan-Boltzmann constant	σ	=	$5.7 \times 10^{-8}\ Wm^{-2}K^{-4}$	5.7×10^{-5} erg $s^{-1}cm^{-2}K^{-4}$
Binding energy of electron in hydrogen ground state		=	13.6 eV	
Bohr magneton	μ_B	=	$e(h/2\pi)/(2mc) = 9.27 \times 10^{-24}$ joules/tesla $= 9.27 \times 10^{-21}$ erg/gauss	
1 atmosphere of pressure		=	$1.013 \times 10^{5}\ newton/m^2$	$1.013 \times 10^{6}\ dynes/cm^2$
Avogadro's number		=		6.025×10^{23} /mole

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INTEGRALS

$\int \sin x \, dx = -\cos x$

$\int \cos x \, dx = \sin x$

$\int x \sin x \, dx = \sin x - x \cos x$

$\int x \cos x \, dx = \cos x + x \sin x$

$\int \sinh x \, dx = \cosh x$

$\int \cosh x \, dx = \sinh x$

DERIVATIVES

$\frac{d}{dx} \sinh x = \cosh x$

$\frac{d}{dx} \cosh x = \sinh x$

$\frac{d}{dx} \tanh x = \mathrm{sech}^2 x$

$\frac{d}{dx} \mathrm{sech}\, x = -\mathrm{sech}\, x \tanh x$

$\frac{d}{dx} \mathrm{csch}\, x = -\mathrm{csch}\, x \coth x$

$\frac{d}{dx} \coth x = -\mathrm{csch}^2 x$

DEFINITE INTEGRALS

$$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

$$\int_{-\infty}^{\infty} x^{2n} e^{-x^2/2a^2} dx = (2\pi a^2)^{1/2} a^{2n} \times 1 \cdot 3 \cdot 5 \cdots (2n-1)$$

SERIES

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)x^2}{2!} \pm \cdots \text{ for } x^2 < 1$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \text{ for } |x| \leq 1$$

$$\log(N!) = N \ln N - N + \ldots\ldots$$

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Gradient	$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$	$\nabla\phi = \frac{\partial\phi}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial\phi}{\partial\varphi}\boldsymbol{\phi} + \frac{\partial\phi}{\partial z}\mathbf{k}$	$\nabla\phi = \frac{\partial\phi}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\boldsymbol{\theta} + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\boldsymbol{\phi}$
Divergence	$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\nabla\cdot\mathbf{A} = \frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}$	$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(A_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$
Curl	$\nabla\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$	$\nabla\times\mathbf{A} = \begin{vmatrix} \frac{1}{r}\mathbf{r} & \boldsymbol{\phi} & \frac{1}{r}\mathbf{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ A_r & rA_\varphi & A_z \end{vmatrix}$	$\nabla\times\mathbf{A} = \begin{vmatrix} \frac{\mathbf{r}}{r^2\sin\theta} & \frac{\boldsymbol{\theta}}{r\sin\theta} & \frac{\boldsymbol{\phi}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ A_r & rA_\theta & rA_\varphi\sin\theta \end{vmatrix}$
Laplacian	$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$	$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2}$	$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\nabla\times\nabla\psi = 0$$

$$\nabla\cdot(\nabla\times\mathbf{a}) = 0$$

$$\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$$

$$\nabla\cdot(\psi\mathbf{a}) = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$$

$$\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a}$$

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$

$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$$

$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}$$

If $\mathbf{x}$ is the coordinate of a point with respect to some origin, with magnitude $r = |\mathbf{x}|$, and $\mathbf{n} = \mathbf{x}/r$ is a unit radial vector, then

$$\nabla\cdot\mathbf{x} = 3 \qquad \nabla\times\mathbf{x} = 0$$

$$\nabla\cdot\mathbf{n} = \frac{2}{r} \qquad \nabla\times\mathbf{n} = 0$$

$$(\mathbf{a}\cdot\nabla)\mathbf{n} = \frac{1}{r}[\mathbf{a} - \mathbf{n}(\mathbf{a}\cdot\mathbf{n})] = \frac{\mathbf{a}_\perp}{r}$$

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CLEBSCH-GORDAN COEFFICIENTS

$j_1 = \frac{1}{2} \qquad j_2 = \frac{1}{2}$:

$$\begin{array}{cc|cccc}
 & & 1 & 1 & 0 & 1 \\
 & & 1 & 0 & 0 & -1 \\
\hline
\frac{1}{2} & \frac{1}{2} & 1 & & & \\
\frac{1}{2} & -\frac{1}{2} & & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & \\
-\frac{1}{2} & \frac{1}{2} & & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & \\
-\frac{1}{2} & -\frac{1}{2} & & & & 1
\end{array}$$

$j_1 = 1 \qquad j_2 = \frac{1}{2}$:

$$\begin{array}{cc|cccccc}
 & & \frac{3}{2} & \frac{3}{2} & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & \frac{3}{2} \\
 & & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} \\
\hline
1 & \frac{1}{2} & 1 & & & & & \\
1 & -\frac{1}{2} & & \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & & & \\
0 & \frac{1}{2} & & \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{3}} & & & \\
0 & -\frac{1}{2} & & & & \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & \\
-1 & \frac{1}{2} & & & & \sqrt{\frac{1}{3}} & -\sqrt{\frac{2}{3}} & \\
-1 & -\frac{1}{2} & & & & & & 1
\end{array}$$

$j_1 = 1 \qquad j_2 = 1$:

$$\begin{array}{cc|ccccccccc}
 & & 2 & 2 & 1 & 2 & 1 & 0 & 2 & 1 & 2 \\
 & & 2 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -2 \\
\hline
1 & 1 & 1 & & & & & & & & \\
1 & 0 & & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & & & & & & \\
0 & 1 & & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & & & & & & \\
1 & -1 & & & & \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{3}} & & & \\
0 & 0 & & & & \sqrt{\frac{2}{3}} & 0 & -\sqrt{\frac{1}{3}} & & & \\
-1 & 1 & & & & \sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{3}} & & & \\
0 & -1 & & & & & & & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & \\
-1 & 0 & & & & & & & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & \\
-1 & -1 & & & & & & & & & 1
\end{array}$$

d FUNCTIONS

$$d^j_{m'm}(\beta) \equiv \langle jm' | \exp(-i\beta J_y/\hbar) | jm \rangle$$

$j = \frac{1}{2}$:

$m' \backslash m$	$+\frac{1}{2}$	$-\frac{1}{2}$
$+\frac{1}{2}$	$\cos \beta/2$	$-\sin \beta/2$
$-\frac{1}{2}$	$\sin \beta/2$	$\cos \beta/2$

$j = 1$:

$m' \backslash m$	$+1$	0	-1
$+1$	$\frac{1}{2}(1+\cos\beta)$	$-\frac{1}{\sqrt{2}}\sin\beta$	$\frac{1}{2}(1-\cos\beta)$
0	$\frac{1}{\sqrt{2}}\sin\beta$	$\cos\beta$	$-\frac{1}{\sqrt{2}}\sin\beta$
-1	$\frac{1}{2}(1-\cos\beta)$	$\frac{1}{\sqrt{2}}\sin\beta$	$\frac{1}{2}(1+\cos\beta)$

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MORE FORMULAS

Normalized wave functions for hydrogen: U_{nlm}

$$U100 = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_o}\right)^{3/2} e^{-Zr/a_o}$$

$$U200 = \frac{1}{\sqrt{32\pi}}\left(\frac{Z}{a_o}\right)^{3/2}\left(2 - \frac{Zr}{a_o}\right)e^{-Zr/2a_o}$$

$$U210 = \frac{1}{\sqrt{32\pi}}\left(\frac{Z}{a_o}\right)^{3/2}\frac{Zr}{a_o}\, e^{-Zr/2a_o}\cos\theta$$

$$U21\pm1 = \frac{1}{\sqrt{64\pi}}\left(\frac{Z}{a_o}\right)^{3/2}\frac{Zr}{a_o}\, e^{-Zr/2a_o}\sin\theta\, e^{\pm i\phi}$$

Where $a_o = \dfrac{4\pi\varepsilon_o\hbar^2}{me^2}\,(1 + m/M)$

$$L_\pm|\ell,m_\ell\rangle = \sqrt{\ell(\ell+1) - m_\ell(m_\ell \pm 1)}\,\hbar\,|\ell,m_\ell \pm 1\rangle$$

$S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ $\quad S_y = \frac{\hbar}{2i}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ $\quad S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ Spin $\frac{1}{2}$ matrices

$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$ $\quad S_y = \frac{\hbar}{\sqrt{2}i}\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 1\\ 0 & -1 & 0\end{pmatrix}$ $\quad S_z = \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$ Spin 1 matrices

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) - \frac{\hat{L}^2(\theta,\phi)\psi}{\hbar^2 r^2}$$ Spherical coordinates

$$E = -\vec{\mu}\cdot\vec{B}$$

$\vec{\mu} = -g_L\mu_B\,\vec{J}/\hbar \qquad g_L = 1 + \dfrac{\hat{\vec{S}}\cdot\hat{\vec{J}}}{J^2}$

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I.1 Radiation by A Charge

The equation of motion of a classical non-relativistic particle, charge q, mass m, including radiation reaction force is

$$m \dot{\vec{v}} = q \left(\vec{E} + \frac{1}{c} \vec{v} \times \vec{B}\right) + m \tau \ddot{\vec{v}} \qquad \text{(eq. 1)}$$

where $\vec{v}$ is velocity in external fields $\vec{E}$, $\vec{B}$ and

$$\tau = \frac{2}{3} \frac{q^2}{mc^3}$$

a) Give a simple (short) derivation of the radiation reaction term starting with the classical expression for the rate at which an accelerated charged particle radiates energy.

b) The non-relativistic charge is placed in the $\vec{E}$ & $\vec{B}$ fields of a plane E.M. wave, in vacuum.

$$\vec{E} = \hat{x} E_o \cos \omega\left(t - \frac{z}{c}\right) \qquad \vec{B} = \hat{y} B_o \cos \omega\left(t - \frac{z}{c}\right)$$

Using conservation of momentum arguments derive an expression for the time averaged radiation force on the charged particle from the E.M. wave, in terms of q, m, E_o, c.

c) Derive the <u>same</u> expression (for the time averaged radiation force on the particle) from part (b) by solving eq. (1) to order ωt and taking time averages of the appropriate forces. (Assume $\omega t \ll 1$.)

NOTE: <u>The three parts of this problem are independent of one another</u>.

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I.2 Motion of a charged particle

Consider the non-relativistic classical motion of a particle of charge q and mass m in a uniform electric field $\vec{E} = E\,\hat{x}$ and a uniform magnetic field $\vec{B} = B\,\hat{y}$. The particle is injected at right angles to these fields with a velocity $\vec{V}_o = V_o\hat{z}$ in the laboratory frame of reference, S.

a) Assuming that E < B (gaussian units) show that it is possible to find a coordinate system S' moving with respect to S in the z direction with speed V' in which the electric field vanishes. Determine V' of S' in terms of E and B.

b) Show that, in S', the particle has a circular orbit. Representing the radius of this orbit in the form

$$R = Ro\,\left(\frac{V_o}{V'} - 1\right)$$

determine Ro in terms of E, B, and M.

c) Show that, in S, the trajectory is spatially periodic. Determine the spatial period Z_o.

d) Without detailed algebraic calculations make a sketch of the particle's trajectory for each of the following cases:

$$V_o/V' = 0,\ 1/2,\ 3/4,\ 1,\ 3/2,\ 2,\ 5/2.$$

In each case make two graphs, showing the motion in the (z,x)-plane in S' and S, respectively. Use Ro as your unit of length.

NOTE: State the units used in your calculations in parts (a) and (b).

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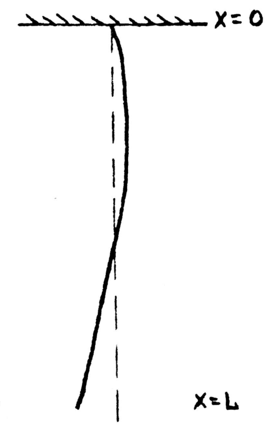
II.1 Hanging Rope

A rope hangs from an overhead support, under gravity. It has length L, mass M. Assume that the mass is uniformly distributed and and the motion is confined to a plane containing the vertical.



a) Set up the equation for small transverse displacement $y(x, t)$ in the rope.

b) State the boundary conditions for the waves at $x = 0$ and $x = L$.

c) Assume a solution for the transverse displacement of form:

$$y(x,t) = U(x)e^{i\omega t},$$

show that two solutions $U_1(x)$ and $U_2(x)$ corresponding to different frequencies ω_1 and ω_2 are orthogonal, i.e.:

$$\int_0^L U_1 U_2 \frac{dx}{G(x)} = 0$$

Find the form of the function $G(x)$.

d) Sketch the function $U_1(x)$ for the lowest mode and estimate the frequency in terms of g, L, M.

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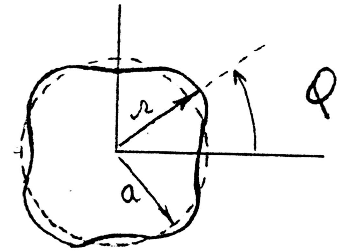
II.2 Drum



Consider an arbitrarily shaped drum with boundary

$$R(\phi) = a[1 + \sum_{p=2}^{\infty} (\varepsilon_p \cos(p\phi) + \bar{\varepsilon}_p \sin(p\phi))]$$

Assume that the coefficients ε_p, $\bar{\varepsilon}_p$, are very small so that the drum is nearly circular.

If the drum head is assumed to be an elastic membrane [with mass density σ = const and surface tension τ = const.] the lagrangian density for a displacement u is given by $L = \frac{1}{2}\sigma (\frac{\partial u}{\partial t})^2 - \frac{1}{2}\tau (\nabla u)^2$

a) Show that the drum's displacement u satisfies the wave equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \nabla^2 u$$

What is the speed of sound c?

Show that the normal mode solutions

$$u(\underline{x},t) = \rho\,(r,\phi)\cos(\omega t)$$

satisfy Helmholtz eq.

$$(\vec{\nabla}^2 + k^2)\,\rho\,(r,\phi) = 0$$

where the wave number is defined by:

$$k = \omega/c$$

b) A general solution to the Helmholtz eq. is

$$\rho\,(r,\phi) = \sum_{m=o}^{\infty} J_m(kr)\,[A_m \cos m\phi + B_m \sin m\phi]$$

Assume that

$A_o \simeq 1$ and that, for $m > 0$, A_m, B_m are of order ε.

Substitute this expression into the boundary condition

$$\rho\,(R\,(\phi),\,\phi) = 0$$

and determine the frequency of the lowest mode and the coefficients A_m, B_m to <u>first order</u> in ε, $\bar{\varepsilon}$. How does the frequency of the lowest mode depend on small deviations from a circular shape?

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III.1 Magnetic Froperties

Consider a solid containing N electrons localized at lattice sites. Let each electron have magnetic moment $\vec{\mu} = g\mu_o \vec{S}$ ($g = 2$, μ_o = Bohr magneton, $\vec{S}$ the spin angular momentum).

a) Suppose that each spin is subject to an effective magnetic fiels H_{eff}. Show that at thermal equilibrium the total magnetic moment M of the solid is given by:

$$M = N\mu_o \tanh\left(\mu_o H_{eff}/kT\right)$$

where k = Boltzmann's constant, and T is the temperature.

b) In the case of a ferromagnetic solid, the effective field H_{eff} acting on a given spin consists of the externally applied field H and a local "Weiss field" ($H_{loc} = qM$) proportional to the alignment of the near neighbor spins. Write down an equation which determines in a self consistent manner the magnetization M.

c) Show that below a certain critical temperature T_c that there is a spontaneous alignment of the spins, and obtain an expression for T_c in terms of the constants defined above.

d) Show that in the vicinity of the critical temperature T_c, but for $T<T_c$ that the magnetization increases with temperature as:

$$\left(\frac{M}{N\mu_o}\right) \sim \sqrt{3} \left(\frac{T_c - T}{T_c}\right)^{1/2}$$

e) Show that the susceptibility $\lim_{H \to 0}(\partial M/\partial H) = \chi$ of the solid diverges as $T \to T_c$ from above in accordance with the Curie-Weiss law

$$\chi \cong \frac{N\mu_o^2}{k(T-T_c)}$$

HINT: $\tanh Z \cong Z - \frac{Z^3}{3}$

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III.2 Boltzmann Gas

a) Starting from the partition function derive the Helmholtz free energy of an ideal Boltzmann gas. Show that it can be written in the following form:

$$F(T,V,N) = -NkT\ln\left(\frac{V}{N}\right) + N\, f\,(T) \qquad \text{-Eq. 1}$$

and determine f(T) which is a function of T only. Explain the physical origin of the so-called "correct Boltzmann counting" which is responsible for the factor NkTln N in (1).

b) Consider a mixture of two ideal Boltzmann gases with N_A and N_B atoms of atom A and Atom B respectively. Since the gases are ideal, $F = F_A + F_B$. Show that this additivity does not hold for the Gibbs free energy which satisfies instead

$$\Phi\,(T,P,N_A,N_B) = \Phi_A(T,P,N_A) + \Phi_B(T,P,N_B) + N_A kT\ln\frac{N_A}{N} + N_B kT\ln\frac{N_B}{N} \qquad \text{-Eq. 2}$$

where $N = N_A + N_B$ and Φ_A, Φ_B are the Gibbs free energy of the pure A and B gas. What is the physical origin of the extra terms in Eq(2)?

c) Consider a dilute solution of common salt in water in which NaCl is fully ionized. Treat the system of water and salt as a mixture of ideal Boltzmann gases. Show that the chemical potential of the water molecule in the solution is given in the limit x << 1 by

$$\mu\,(T,P) = \mu_o(T,P) - 2\,kTX$$

where $X = N_{NaCl}/N_{water}$ is the concentration of salt and $\mu_o(T,P)$ is the chemical potential of pure water.

d) Show that the freezing point of the solution is depressed from that of pure water by the amount

$$\Delta T = 2\,X\,\frac{RT_o^{\,2}}{\Delta H}$$

where ΔH = 1436 cal/mole is the heat of melting of ice and T_o = 273 K is the melting temperature of ice. R is the Boltzmann constant per mole and equals 1.987 cal/K-mole.

Evaluate ΔT when X = 1%.

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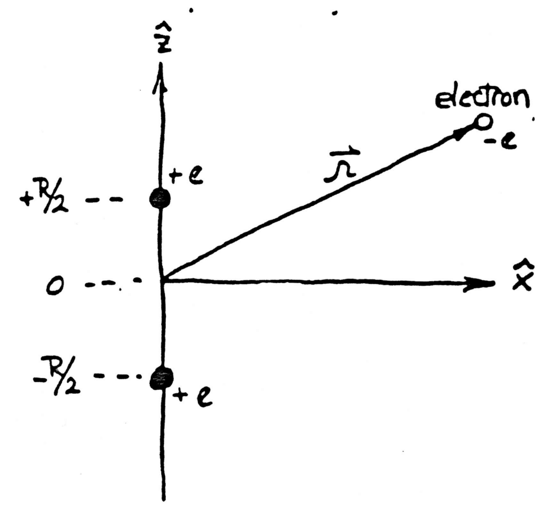
IV.1 Hydrogen Molecular Ion

Estimate the binding energy of the H_2^+ ion with a simple variational calculation. Assume the nuclei are infinitely heavy and are separated by distance R.



$$H = -\frac{\vec{\nabla}^2}{2m} - \frac{e^2}{|\vec{r} - \vec{R}/2|} - \frac{e^2}{|\vec{r} + \vec{R}/2|} + \frac{e^2}{R}$$

Use the normalized Gaussian trial wave function

$$\psi(r) = \left|\frac{8\alpha^3}{\pi^3}\right|^{1/4} e^{-\alpha r^2} \quad , \alpha = \text{parameter}$$

In the $R \to \infty$ limit use this wave function to estimate the binding energy of an H atom (center ψ on one of the nuclei). Use atomic units $\hbar = m = e^2 = 1$

$$H\ \text{atom} = \frac{\nabla^2}{2} - \frac{1}{r}$$

Answer:

$E = -4/3\pi = -0.424$

$\alpha = 8/9\pi$

Note: There are some integrals tabulated at the end of the problem

Discuss (1) How α was determined and what it means.

(2) How this energy compares to the exact energy of an H atom $E = -0.5$. Why is there a difference?

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Now for the H_2^+ molecule, use the same trial wave function (centered between the two nuclei) to estimate the binding energy.

Hint: The potential energy between the electron and one of the proton is

$$V = - \frac{d^3 r}{\left| \vec{r} - \frac{R}{2} \hat{z} \right|} e^{-2 \alpha r^2} \frac{8\alpha^3}{\pi^3}$$

The following integral may be useful

$$I = \int_0^\infty x \, d x \int_{-1}^{1} d(\cos \theta) e^{-2 \alpha \left(x^2 + \frac{R^2}{4} + Rx \cos \theta\right)} = \frac{\pi}{8\alpha} \frac{1}{R\alpha} \operatorname{erf} \left(\frac{\alpha}{2} R \right)$$

The error function is:

$$\operatorname{erf}(x) = \frac{2}{\pi} \int_0^x e^{-t^2} dt$$

which has the limits
$$\operatorname{erf}(o) = 0$$
$$\operatorname{erf}(\infty) = 1$$

For this wave function show the energy is

$$E(R,x) = \frac{3}{2} \alpha - \frac{4}{R} \operatorname{erf} \left(\frac{\alpha}{2} R \right) + \frac{1}{R}$$

Discuss the meaning of each of three terms

What is the limit of the 2nd term as $\alpha \to \infty$? Why?

This is a function of two variables. How would one determine the equlibrium inter-nuclear separation (neglect nuclear vibrations)?

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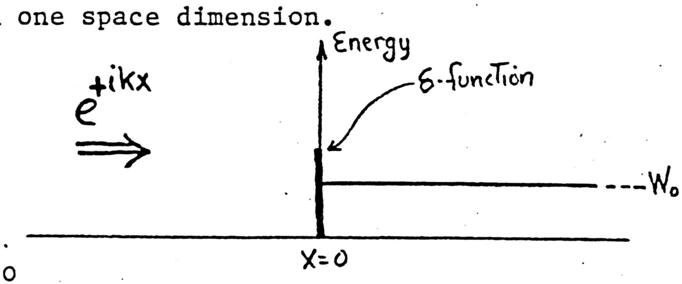
IV.2 One Dimensional Scattering

2. Scattering from a step function and a delta function.

Consider a beam of particles of mass, m and momentum, k, incident from the left on the following potential in one space dimension.



$$V(x) = V_o\,\delta(x) + W_o\,\Theta(x)$$

$$\Theta(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geqslant 0 \end{cases}$$

Assume the incident energy is greater than W_o and that V_o, W_o are greater than zero. Use units $h = c = 1$.

Calculate the ratios of incident flux to transmitted flux in region II (T) and reflected flux (traveling to the left) in region I (R).

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Massachusetts Institute of Technology

Department of Physics

February 11, 1986

Doctoral General Examination

Part I

Five Hours

Instructions

This examination is divided into four groups of 60 points each. USE A SEPARATE EXAMINATION BOOKLET FOR EACH GROUP. The number of points is the same for all the problems -- 15 points each. BE SURE TO LIST THE PROBLEM NUMBER WITH EACH SOLUTION.

A diagram or sketch as part of the answer is often useful, particularly when a problem asks for a qualitative response.

NO books or reference materials may be used.

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I-1. Physical Magnitudes

Give approximate numerical values for the following quantities. The answers may be in any appropriate units, but they should be specified.

1. Bohr radius of the hydrogen atom.
2. Work function of a typical metal.
3. Rest energy of a π° meson.
4. Binding energy per nucleon of iron (Fe).
5. Energy of a Lyman α photon.
6. Surface temperature of the sun.
7. Magnetic field at the earth's equator.
8. Nuclear magnetic moment of helium-4 (He^4).
9. Distance to the sun.
10. Range of wavelengths in visible light.
11. The difference in rest energy between a proton and a neutron.
12. K-shell binding energy of lead (Pb).
13. Index of refraction of water.
14. Shortest orbital period for a small satellite of the earth.
15. Diffraction-limited angular resolution of the human eye.

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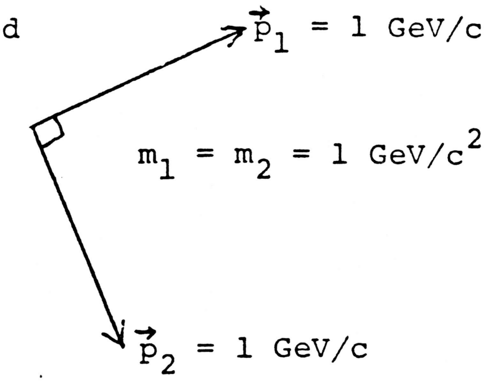
I-2. Relativistic Dynamics

Two particles are observed to emerge from a nuclear interaction. Their velocities are at right angles to each other. Each particle has a rest mass of 1.0 GeV/c^2 and a momentum of magnitude 1.0 GeV/c. It is hypothesized that the two particles are the only decay products of a single short-lived particle. What would have been the rest mass of the parent particle?



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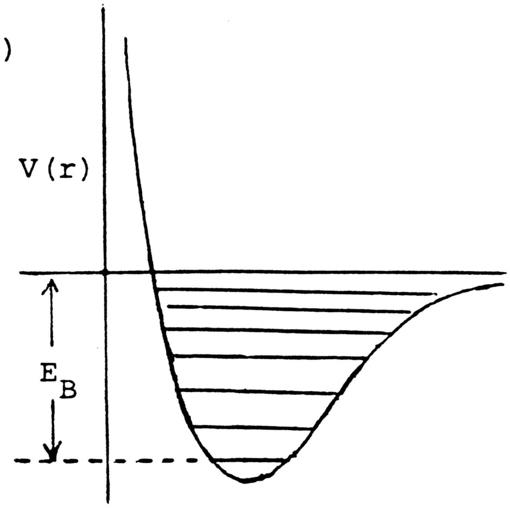
I-3. Molecular Hydrogen and Deuterium

The molecular vibrations in Hydrogen (H_2) and Deuterium (D_2) are determined by identical potential energy curves V(r), as functions of the nuclear separation r. For H_2, the dissociation energy E_B is 4.476 eV, and the vibrational energy $\hbar\omega$ is 0.545 eV.

Obtain the dissociation energy for the D_2 molecule.



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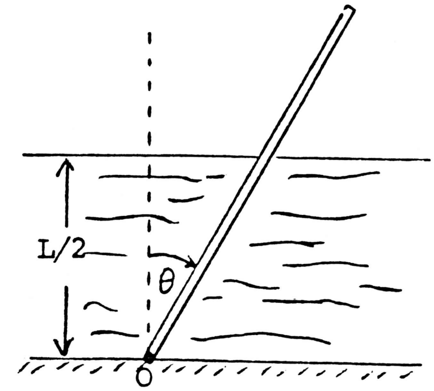
I-4. Floating Rod



A thin wooden rod of length L is pivoted about a horizontal axis through the point O at the bottom of a tank of water of depth L/2. The density of the wood is half the density of the water.

a) Show that the rod has an orientation with stable equilibrium in the range $0 < \theta < 60°$, and find the equilibrium angle θ_o.

b) Ignoring motions of the water, calculate the period of small angular oscillations about the equilibrium position.

II-1. Spin-orbit Splitting

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The spin-orbit energy of the valence electron of an alkali atom, in a state (n, ℓ), can be expressed as

$$V_{s-o} = C_{n,\ell}\,(\vec{\ell}\cdot\vec{s})$$

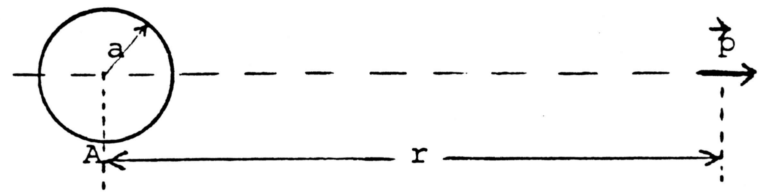
a) Obtain an expression for the energy difference between the states $(n,\ell,j=\ell+\frac{1}{2})$ and $(n,\ell,j=\ell-\frac{1}{2})$.

b) Explain in qualitative terms the physical origin of the spin-orbit interaction.

MIT

II-2. Dipole and Conducting Sphere

An electric dipole of moment $\vec{p}$ is placed at a distance r from a perfectly conducting, isolated sphere of radius a. The dipole points radially away from the sphere, as shown. Assume that $r \gg a$ and that the sphere carries no net charge.



a) What are the boundary conditions on the $\vec{E}$ field at the surface of the sphere?

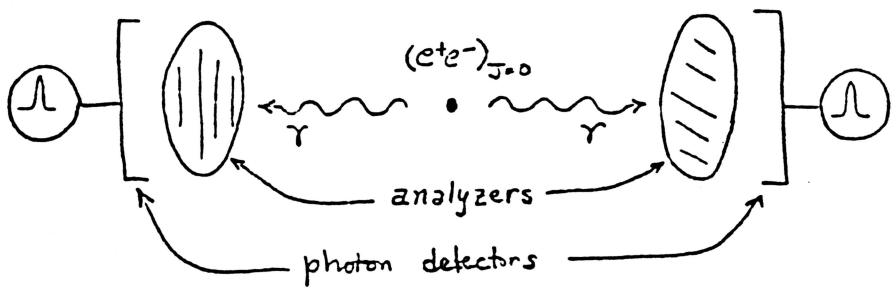
b) By applying the boundary conditions at point A on the sphere, find the approximate value of the induced dipole moment, and hence the force exerted on the external dipole. [It is not necessary to solve Laplace's equation for this system in detail.]

MIT

I-3. Polarized Photons from Positronium

Singlet positronium decays by emitting two oppositely directed photons that are polarized at right angles with respect to each other.

An experiment is performed with photon detectors behind polarization analyzers, as shown. Each analyzer has a preferred axis, such that light polarized in that direction is transmitted perfectly, while light polarized in the perpendicular direction is completely absorbed. The analyzer axes are at right angles to each other.



When very many events are observed, what is the ratio of the number of events in which both detectors record a photon to the number in which only one detector records a photon?

MIT

II-4. Defects in a Solid

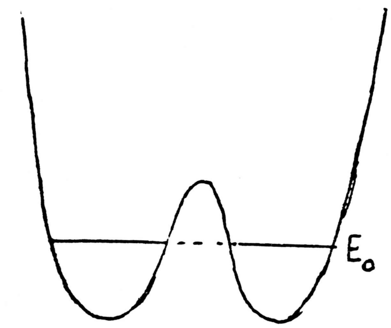
A crystalline solid contains N similar, immobile, statistically independent defects. Each defect has 5 possible states $\psi_1, \psi_2, \psi_3, \psi_4$ and ψ_5 with energies $\varepsilon_1 = \varepsilon_2 = 0$, $\varepsilon_3 = \varepsilon_4 = \varepsilon_5 = \Delta$.

Find the defect contribution to the entropy of the crystal as a function of Δ and the temperature T.

MIT

III-1. Double Well

The lowest energy level in the symmetrical one-dimensional potential sketched here is positioned as shown.

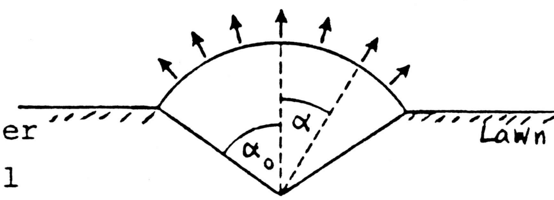


a) Indicate the approximate position of the energy of the first excited state. Explain the basis of your choice.

b) Make sketches of the wave functions for these two lowest states.

MIT

III-2. Lawn Sprinkler



A lawn sprayer consists of a spherical cap ($\alpha_0 = 45^\circ$) pierced with a large number of equal holes through which water is ejected with speed v_0. The lawn will not be uniformly sprayed if these holes are evenly spaced. How must $\rho(\alpha)$, the number of holes per unit area of the cap, be chosen to achieve uniform spraying of a circular area around the cap? Assume that the radius of the sprinkling cap is very much less than the radius of the area to be sprayed, and that the surface of the cap is at the level of the lawn. [Ignore any effects of air resistance.]

MIT

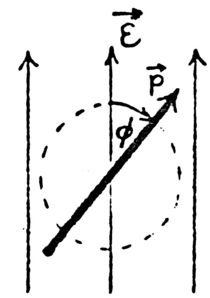
III-3. The Big Bang

Early in the evolution of the universe, when the universe was extremely hot and dense, matter and radiation were in thermal equilibrium. However, when the temperature fell to about 3000 K, matter and the cosmic background radiation became decoupled, and it is assumed that they have remained decoupled throughout the subsequent expansion. The temperature of the cosmic (black body) radiation has been measured to be about 3 K now. Assuming adiabatic expansion, by what factor has the universe increased in volume since the decoupling of cosmic radiation and matter? [The Helmholtz free energy of thermal radiation is given by

$$F(T,V) = -\frac{1}{45}\frac{\pi^2}{c^3\hbar^3}(kT)^4 V.]$$

MIT

III-4. Quantum Rotator



A rigid rotator with electric dipole moment $\vec{p}$ is confined to rotate in a plane. The rotator has moment of inertia I about the (fixed) rotation axis. A weak uniform electric field $\vec{\mathcal{E}}$ lies in the rotation plane.

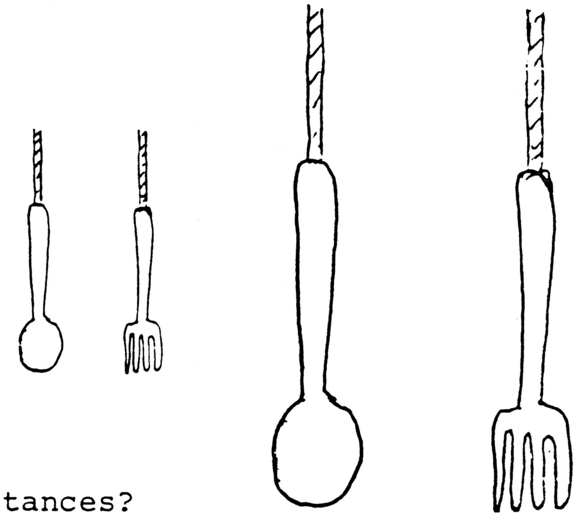
a) Write down the Hamiltonian for this system.

b) Using second-order perturbation theory, calculate to order $\mathcal{E}^2$ the effect of the electric field on the energy of the lowest state of this system.

MIT

IV-1. Capacitors

A metal spoon and fork, mounted on insulating rods, form a capacitor. Another such capacitor is made from a larger spoon and fork, such that all linear dimensions, including spacings, are bigger by a factor n than for the first system.



a) What is the ratio of the capacitances? Give a clear account of your reasoning.

b) If the potential difference between spoon and fork is given the same value in both cases, compare the surface charge densities at corresponding points.

c) If the insulation of the air breaks down at a certain field strength, what is the ratio of the maximum stored energies for the two systems?

MIT

IV-2. Ideal Gas Mixture

A container of volume V encloses a mixture of two monatomic ideal gases consisting of N_1 atoms of one type and N_2 atoms of another type at temperature T.

a) What is the pressure P of this gas?

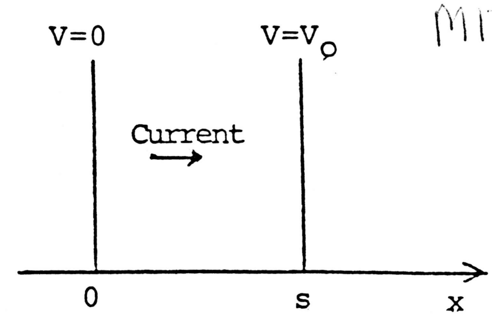
The gas is now compressed slowly to one-third of its original volume. If the container is kept thermally insulated:

b) What is the final temperature of the gas?

c) What is the final pressure?

IV-3. <u>Vacuum Diode</u>

MIT



The sketch represents a plane-parallel vacuum diode that is being operated under space-charge-limited conditions. It is found that the charge density in the gap is given by

$$\rho(x) = -K\,x^{-2/3},$$

where K is a positive constant and x is the position within the gap measured from the cathode.

a) Find the potential V(x) in the gap between cathode and anode; assume that the electric field vanishes at x = 0.

b) Assuming that the electrons leave the cathode with negligible velocity, find their subsequent velocity at any point as a function of x, V_o, m and the electrode spacing s.

MIT

IV-4. Relativistic Rocket

A rocket starts from rest on earth and travels along a straight path toward a distant star. Throughout the trip the rocket maintains a constant acceleration equal to g as measured in its own instantaneous rest-frame.

Show tnat, for any value of its velocity v, the rocket's instantaneous acceleration as measured in the earth's frame of reference is given by $g/[\gamma(v)]^3$, where $\gamma(v) = (1 - v^2/c^2)^{-\frac{1}{2}}$.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Department of Physics

February 11, 1986

DOCTORAL GENERAL EXAMINATION

PART II

Five Hours

INSTRUCTIONS

This examination consists of four sections with two problems each. SOLVE ONE PROBLEM FROM EACH GROUP OF THE FOUR SECTIONS (a total of four problems). It is advisable to read carefully both problems in each section before making your choice. The problems in each section have been chosen to be of comparable difficulty. USE A SEPARATE FOLD OF PAPER FOR EACH PROBLEM AND LABEL IT WITH THE PROBLEM NUMBER AND YOUR NAME.

NO books or references may be used.

You may find some of the information on the next pages useful.

MIT

Constant	Symbol		MKS	cgs
Boltzmann constant	k	=	$1.38\text{x}10^{-23}J/^{o}K$	$1.38\text{x}10^{-16}erg/^{o}K$
Electron mass	m_e	=	$9.1\text{x}10^{-31}$ kg	$9.1\text{x}10^{-28}$ g
Proton mass	m_p	=	$1.67\text{x}10^{-27}$ kdg	$1.67\text{x}10^{-24}$ g
Gravitational constant	G	=	$6.67\text{x}10^{-11}$ $N\text{-}m^2/kg^2$	$6.67\text{x}10^{-8}$ $dyne\text{-}cm^2/g^2$
Planck's constant	$h/2\pi$	=	$1.05\text{x}10^{-34}$ J-s	$1.05\text{x}10^{-27}$ erg-sec
			$h/2\pi=6.58\text{x}10^{22}$ MeV sec	
Speed of light	c	=	$3.0\text{x}10^{8}$ m/sec	$3.0\text{x}10^{10}$ cm/sec
			$hc/2\pi=1.97\text{x}10^{-11}$ MeV cm	
Electronic charge	e	=	$1.6\text{x}10^{-19}$ coul	$4.8\text{x}10^{-10}$ esu
Energy units	1 eV	=	$1.6\text{x}10^{-19}$ J	$1/6\text{x}10^{-12}$ erg
Permittivity of vacuum	ε_0	=	$8.85\text{x}10^{-12}$ farad/m	
Permeability of vacuum	μ_0	=	$1.26\text{x}10^{-6}$ henry/m	
Acceleration of gravity	g	=	$9.8\ m/sec^2$	$980\ cm/sec^2$
Stefan-Boltzmann constant	σ	=	$5.7\text{x}10^{-8}\ Wm^{-2}K^{-4}$	$5.7\text{x}10^{-5}$ erg $s^{-1}cm^{-2}K^{-4}$
Binding energy of electron in hydrogen ground state		=	13.6 eV	
Bohr magneton	μ_B	=	$e(h/2\pi)/(2mc)=9.27\text{x}10^{-24}$ joules/tesla	
			$=9.27\text{x}10^{-21}$ erg/gauss	
1 atmosphere of pressure		=	1.013×10^5 newton/m^2	1.013×10^6 dynes/cm^2
Avogadro's number		=		6.025×10^{23} /mole

MIT

INTEGRALS

$\int \sin x \, dx = -\cos x$

$\int \cos x \, dx = \sin x$

$\int x \sin x \, dx = \sin x - x \cos x$

$\int x \cos x \, dx = \cos x + x \sin x$

$\int \sinh x \, dx = \cosh x$

$\int \cosh x \, dx = \sinh x$

DERIVATIVES

$\frac{d}{dx} \sinh x = \cosh x$

$\frac{d}{dx} \cosh x = \sinh x$

$\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$

$\frac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \tanh x$

$\frac{d}{dx} \operatorname{csch} x = -\operatorname{csch} x \coth x$

$\frac{d}{dx} \coth x = -\operatorname{csch}^2 x$

DEFINITE INTEGRALS

$$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

$$\int_{-\infty}^{\infty} x^{2n} e^{-x^2/2a^2} dx = (2\pi a^2)^{1/2} a^{2n} \times 1\cdot 3\cdot 5 \cdots (2n-1)$$

SERIES

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)x^2}{2!} \pm \cdots \text{ for } x^2 < 1$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \text{ for } |x| \le 1$$

$$\log(N!) = N \ln N - N + \ldots\ldots$$

MIT

FORMULAS OF VECTOR ANALYSIS

	Rectangular coordinates	Cylindrical coordinates	Spherical coordinates
Conversion to rectangular coordinates		$x = r\cos\varphi \quad y = r\sin\varphi \quad z = z$	$x = r\cos\varphi\sin\theta \quad y = r\sin\varphi\sin\theta$ $z = r\cos\theta$
Gradient.	$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$	$\nabla\phi = \frac{\partial\phi}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial\phi}{\partial\varphi}\boldsymbol{\phi} + \frac{\partial\phi}{\partial z}\mathbf{k}$	$\nabla\phi = \frac{\partial\phi}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\boldsymbol{\theta} + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\boldsymbol{\phi}$
Divergence.	$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\nabla\cdot\mathbf{A} = \frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}$	$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(A_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$
Curl.	$\nabla\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$	$\nabla\times\mathbf{A} = \begin{vmatrix} \frac{1}{r}\mathbf{r} & \boldsymbol{\phi} & \frac{1}{r}\mathbf{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ A_r & rA_\varphi & A_z \end{vmatrix}$	$\nabla\times\mathbf{A} = \begin{vmatrix} \frac{\mathbf{r}}{r^2\sin\theta} & \frac{\boldsymbol{\theta}}{r\sin\theta} & \frac{\boldsymbol{\phi}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ A_r & rA_\theta & rA_\varphi\sin\theta \end{vmatrix}$
Laplacian.	$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$	$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2}$	$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\nabla\times\nabla\psi = 0$$

$$\nabla\cdot(\nabla\times\mathbf{a}) = 0$$

$$\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$$

$$\nabla\cdot(\psi\mathbf{a}) = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$$

$$\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a}$$

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$

$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$$

$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}$$

If $\mathbf{x}$ is the coordinate of a point with respect to some origin, with magnitude $r=|\mathbf{x}|$, and $\mathbf{n}=\mathbf{x}/r$ is a unit radial vector, then

$$\nabla\cdot\mathbf{x} = 3 \qquad \nabla\times\mathbf{x} = 0$$

$$\nabla\cdot\mathbf{n} = \frac{2}{r} \qquad \nabla\times\mathbf{n} = 0$$

$$(\mathbf{a}\cdot\nabla)\mathbf{n} = \frac{1}{r}[\mathbf{a} - \mathbf{n}(\mathbf{a}\cdot\mathbf{n})] \equiv \frac{\mathbf{a}_\perp}{r}$$

MIT

CLEBSCH-GORDAN COEFFICIENTS

$j_1 = \frac{1}{2} \qquad j_2 = \frac{1}{2}$:

$$\begin{array}{cc|cccc}
 & & 1 & 1 & 0 & 1 \\
 & & 1 & 0 & 0 & -1 \\
\hline
\frac{1}{2} & \frac{1}{2} & 1 & & & \\
\frac{1}{2} & -\frac{1}{2} & & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & \\
-\frac{1}{2} & \frac{1}{2} & & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & \\
-\frac{1}{2} & -\frac{1}{2} & & & & 1
\end{array}$$

$j_1 = 1 \qquad j_2 = \frac{1}{2}$:

$$\begin{array}{cc|cccccc}
 & & \frac{3}{2} & \frac{3}{2} & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & \frac{3}{2} \\
 & & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} \\
\hline
1 & \frac{1}{2} & 1 & & & & & \\
1 & -\frac{1}{2} & & \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & & & \\
0 & \frac{1}{2} & & \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{3}} & & & \\
0 & -\frac{1}{2} & & & & \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & \\
-1 & \frac{1}{2} & & & & \sqrt{\frac{1}{3}} & -\sqrt{\frac{2}{3}} & \\
-1 & -\frac{1}{2} & & & & & & 1
\end{array}$$

$j_1 = 1 \qquad j_2 = 1$:

$$\begin{array}{cc|ccccccccc}
 & & 2 & 2 & 1 & 2 & 1 & 0 & 2 & 1 & 2 \\
 & & 2 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -2 \\
\hline
1 & 1 & 1 & & & & & & & & \\
1 & 0 & & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & & & & & & \\
0 & 1 & & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & & & & & & \\
1 & -1 & & & & \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{3}} & & & \\
0 & 0 & & & & \sqrt{\frac{2}{3}} & 0 & -\sqrt{\frac{1}{3}} & & & \\
-1 & 1 & & & & \sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{3}} & & & \\
0 & -1 & & & & & & & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & \\
-1 & 0 & & & & & & & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & \\
-1 & -1 & & & & & & & & & 1
\end{array}$$

d FUNCTIONS

$$d^{j}_{m'm}(\beta) \equiv \langle jm' | \exp(-i\beta J_y/\hbar) | jm \rangle$$

$j = \frac{1}{2}$:

$m' \backslash m$	$+\frac{1}{2}$	$-\frac{1}{2}$
$+\frac{1}{2}$	$\cos \beta/2$	$-\sin \beta/2$
$-\frac{1}{2}$	$\sin \beta/2$	$\cos \beta/2$

$j = 1$:

$m' \backslash m$	$+1$	0	-1
$+1$	$\frac{1}{2}(1 + \cos \beta)$	$-\frac{1}{\sqrt{2}} \sin \beta$	$\frac{1}{2}(1 - \cos \beta)$
0	$\frac{1}{\sqrt{2}} \sin \beta$	$\cos \beta$	$-\frac{1}{\sqrt{2}} \sin \beta$
-1	$\frac{1}{2}(1 - \cos \beta)$	$\frac{1}{\sqrt{2}} \sin \beta$	$\frac{1}{2}(1 + \cos \beta)$

MIT

MORE FORMULAS

Normalized wave functions for hydrogen: U_{nlm}

$$U100 = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_o}\right)^{3/2} e^{-Zr/a_o}$$

$$U200 = \frac{1}{\sqrt{32\pi}}\left(\frac{Z}{a_o}\right)^{3/2}\left(2 - \frac{Zr}{a_o}\right)e^{-Zr/2a_o}$$

$$U210 = \frac{1}{\sqrt{32\pi}}\left(\frac{Z}{a_o}\right)^{3/2}\frac{Zr}{a_o}\, e^{-Zr/2a_o}\cos\theta$$

$$U21 \pm 1 = \frac{1}{\sqrt{64\pi}}\left(\frac{Z}{a_o}\right)^{3/2}\frac{Zr}{a_o}\, e^{-Zr/2a_o}\sin\theta\, e^{\pm i\phi}$$

Where $a_o = \frac{4\pi\varepsilon_o\hbar^2}{me^2}(1 + m/M)$

$$L_\pm|\ell,m_\ell> = \sqrt{\ell(\ell+1) - m_\ell(m_\ell \pm 1)}\,\hbar|\ell,m_\ell \pm 1>$$

$S_x = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}$ $S_y = \frac{\hbar}{2i}\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ $S_z = \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ Spin $\frac{1}{2}$ matrices

$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}$ $S_y = \frac{\hbar}{\sqrt{2}i}\begin{pmatrix}0&1&0\\-1&0&1\\0&-1&0\end{pmatrix}$ $S_z = \hbar\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$ Spin 1 matrices

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) - \frac{\hat{L}^2(\theta,\phi)\psi}{\hbar^2 r^2}$$ Spherical coordinates

$$E = -\vec{\mu}\cdot\vec{B}$$

$$\vec{\mu} = -g_L\mu_B\vec{J}/\hbar \qquad g_L = 1 + \frac{\hat{\vec{S}}\cdot\hat{\vec{J}}}{J^2}$$

MIT

I. 1

Classical Mechanics

Problem #1: Adiabatic Invariants.

If the parameters of a system are changed slowly, the energy may change, however the energy divided by a characteristic frequency often does not. This adiabatic invariant, which is nearly constant in classical physics, is a natural thing to quantize when making a transition to quantum physics.

Consider a particle moving in a one dimensional box with rigid walls. Let the righthand wall move away from the other wall at a constant velocity V_w. At $\tau = 0$ the walls are a distance ℓ apart and the particle is moving with velocity V.

(a) What is the restriction on the wall's velocity V_w so that the wall's motion represents an adiabatic change in the system?

(b) Calculate how the kinetic energy of the particle E(t), and a characteristic frequency of its motion $\nu(t)$, change with time.

(c) Show that the adiabatic invariant

$$\frac{E(t)}{\nu(t)}$$ is indeed constant with time.

MIT

I.2

Classical Mechanics

#2: Getting to the sun.

Assume one has a spacecraft (which has just escaped from the earth) in circular orbit about the sun one astronomical unit away (1 A.U. = radius of earth's orbit). One way to get to the sun is to fire a rocket to stop the orbital motion and just fall in.

(a) What is the required velocity change ΔV provided by the rocket?

(b) Show that it takes $\frac{1}{4\sqrt{2}}$ years to fall in.

Hint: The following integral

$$\int \frac{x^2 dx}{\sqrt{\alpha - x^2}} = -\frac{x}{2}\sqrt{\alpha - x^2} + \frac{\alpha}{2} \sin^{-1}\left[\frac{x}{\sqrt{\alpha}}\right]$$

may be useful where $x = r^{1/2}$.

Alternatively, one could fire the rocket forward to increase the orbital velocity to solar escape velocity. Then when one is far from the sun and moving slowly a small rocket burn could send you falling all the way back into the sun.

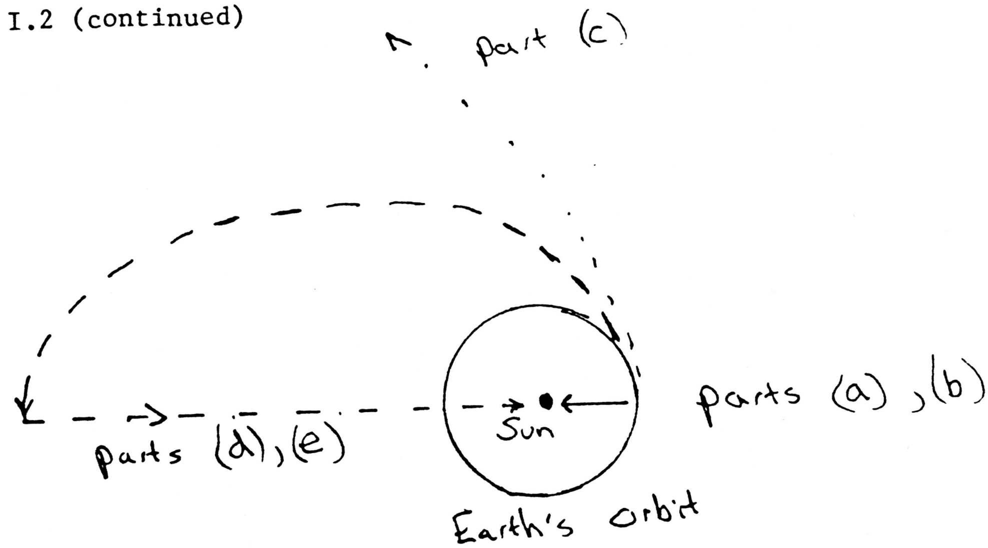
(c) What is the required total ΔV of the rocket for this route? [Ignore the other planets].

A practical trade-off between rocket size and travel time is to boost into an eliptical orbit with an apogee of, say, 5 A.U., then stop dead at apogee and fall in.

(d) What is the <u>total</u> ΔV for this course?

(e) Can you calculate how long it would take to get to the sun this way? [At least provide an estimate.]

(cont.)



Note, the solar polar mission will fly this last course, but it will use Jupiter's gravity instead of a rocket burn at apogee.

MIT

SECTION II: ELECTRODYNAMICS

II.1

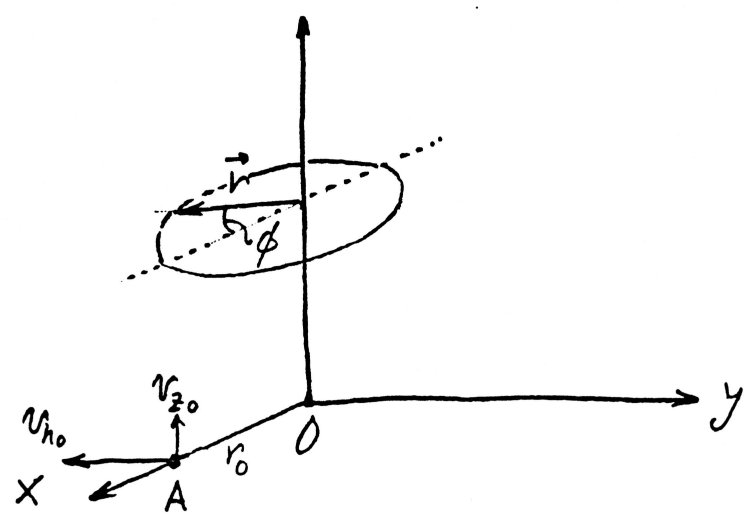
The Magnetic Mirror or the Magnetic Bottle

A. Consider a magnetic field $\vec{B}$ which is axially symmetrical about the z axis. Let the z component of the field increase linearly with z in accordance with

$$B_z(z) = B_o + B_o' z.$$

a) Find the radial component $B_r\ (r,z)$ as a function of z and r.

The coordinate axes are shown below:



B. Let a particle with mass m, charge + q be injected into the field at z = 0 (x y plane) at point A, a distance r_o from 0 with a velocity in the x y plane given by

$$V_{ho} = \frac{q}{mc} B_o r_o$$

Also, let the z component of the velocity of this particle at z=0 be V_{zo}. Let the motion be non-relativistic.

a) Show that throughout the subsequent motion

$$v^2(t) = v_{ho}^2 + v_{zo}^2 = v_o^2$$

b) Obtain the equation of motion for the time dependence of the x y component $(\vec{v}_h)$ of the particle motion.

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II.1 (continued)

c) Use your result in (b) to obtain an equation for the dependence of $|v_h|$ on z.

d) Assume that during each orbit around the z axis the size of the orbit changes a negligible amount compared to the orbit size. Using this fact show that the magnitude of the horizontal velocity varies with z as

$$\left(\frac{v_h(z)}{v_{h_o}}\right) = \sqrt{\left(\frac{B(z)}{B(0)}\right)}$$

e) Obtain the value of z at which $v_z = 0$. This is the reflection point of this magnetic mirror. Here the spiraling particle stops its upward motion and starts moving downward toward z = 0.

f) Find the radius of the orbit at this reflection point. Compare with the original radius.

Express your answers to (e) and (f) entirely in terms of B_o, B_o', v_{zo} and v_{ho}.

Remember that in cylindrical coordindates

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\phi A_z}{\partial z}$$

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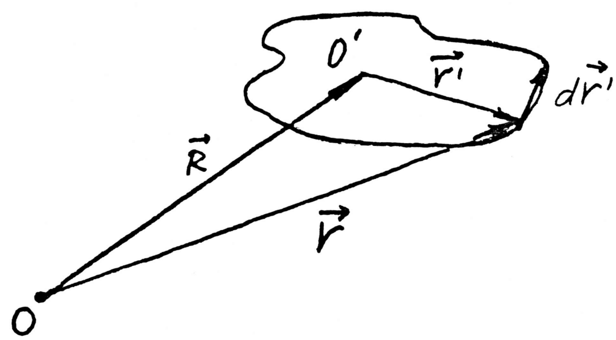
SECTION II: ELECTRODYNAMICS

II.2

A) Consider a small, stationary loop of arbitrary shape carrying a fixed current I (see diagram below).



O$'$ is a reference point defining the position of the loop.

Let the loop be in a magnetic field $\vec{B}(\vec{r})$ whose magnitude and direction is some slowly varying function of position $\vec{r}$ only. Start from the Lorentz force and show that the total force $\vec{F}$ on the loop is given by an expression of the form

$$\vec{F} = \vec{\nabla}_R \left(\vec{M}\cdot\vec{B}(\vec{R}) \right)$$

Give an expression for $\vec{M}$ and identify its meaning.

B) Consider an arbitrarily shaped small loop which is being pulled with constant velocity $\vec{V}$ through the same magnetic field $\vec{B}(\vec{r})$ as in (A). Calculate the E.M.F. induced in the loop in terms of the geometric quantities $\vec{M}/I$ and $\vec{B}(\vec{R})$.

C) Show that the mechanical power needed to pull the loop with velocity $\vec{V}$ is exactly equal to the electrical power dissipation in the loop. Assume the loop has an ohmic resistance R sufficiently large to neglect the effect of self inductance.

Hint...remember the Vector identity.

$$d\vec{r}\,(\vec{r}\cdot\vec{a}) = 1/2\, d\left(\vec{r}\,(\vec{r}\cdot\vec{a})\right) + 1/2\,(\vec{r} \times d\vec{r}\,) \times \vec{a}$$

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SECTION III: STATISTICAL MECHANICS/THERMODYNAMICS

III.1

Consider the Einstein model for lattice vibrations in a crystal. Each of the N constituent atoms undergoes three dimensional harmonic oscillations with the same frequency ω about its binding site. Each atom is bound to its lattice position with energy $-\varepsilon_B$. (Include zero point vibrations.)

a) Show that the partition function in the canonical ensemble for the entire crystal is given by

$$Z = e^{N\beta\varepsilon_B}(2 \sinh \beta h\omega/2)^{-3N}$$

where $\beta = 1/kT$.

b) Obtain the Helmholtz free energy F.

c) Obtain the mean energy $\bar{E}$ of the crystal valid at arbitrary temperature T.

i) Show that for $kT \ll h\omega$ your result property describes the zero point energy of the vibrating atoms.

ii) Show that for $kT \gg h\omega$ your result is exactly in agreement with the equipartition theorem of classical statistical mechanics.

d) Obtain the Equation of State, ie. the pressure (p) volume (V), and temperature (T) relationship for this solid. In carrying out your calculation define $\phi(V) = -N\varepsilon_B(V)$ as the volume dependent cohesive energy. Also, keep in mind that upon changing the volume of the solid, the vibration frequency ω changes in accordance with the definition of Gruneisen's constant

$$\gamma = -(\partial \ln\omega/\partial \ln V) = \text{constant}$$

Express your result for the equation of state in terms of γ, V, ϕ, and the vibrational energy $U = (\bar{E} - \phi)$ obtained in Part c above.

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SECTION III: STATISTICAL MECHANICS/THERMODYNAMICS

III.2

Consider a system of N non interacting distinguishable particles. Each particle can be in one or two states whose energies are $+\epsilon_0$ and $-\epsilon_0$.

a) Obtain an expression for mean energy ($\bar{E}$) of the system in equilibrium at temperature T. Express your answer in terms of ε_0, N and $\beta = 1/kT$.

b) Obtain the values of $\bar{E}$ in the limits $kT \ll \varepsilon_0$ and $kT \gg \varepsilon_0$.

c) Obtain a formula for the specific heat C_V, valid for all values of $\beta\varepsilon_0$.

d) Show that $C_V \to 0$ in both the limits $\beta\varepsilon_0 \gg 1$ (low temperature), and $\beta\varepsilon_0 \ll 1$ (high temperature). Explain physically the reason that the specific heat goes to zero in both these cases.

e) Obtain an equation for the value of $\beta\varepsilon_0$ for which the specific heat is maximum. Make an approximate numerical estimate for the value of $\beta\varepsilon_0$ at which this maximum occurs. Explain physically why C_V is maximized there.

IV.1 MTT

Quantum Mechanics

Problem #1: Nuclear matter in a mean field approximation.

Consider a uniform system of N nucleons of mass M moving in a large volume V. Assume that the nucleons interact with a two-body potential, v(r), which is a function only of the distance, r, between the two nucleons. The Hamiltonian is

$$H = \sum_i \frac{-\hbar^2 \nabla_i^2}{2M} + \sum_{i<j} V(\overrightarrow{r_i - r_j})$$

(a) Calculate the (Hartree) expectation value of the potential energy per particle for a many-body wavefunction which is constant over the volume V (ignore the required antisymmetry).

$$\langle V \rangle = \int \psi^* \sum_{i<j} v(\vec{r}_i - \vec{r}_j)\, d\tau$$

$$\psi = \text{constant}$$

(b) Estimate the kinetic energy per particle assuming a free Fermi gas with each momentum state occupied by four particles [spin up, spin down, neutrons and protons].

(c) Calculate and plot qualitatively the total energy per particle vrs density. Assume the volume integral of the potential, $\int v(r) d^3r$, is finite and attractive.

Does this energy vrs density curve describe the equilibrium properties of nuclear matter, explain?

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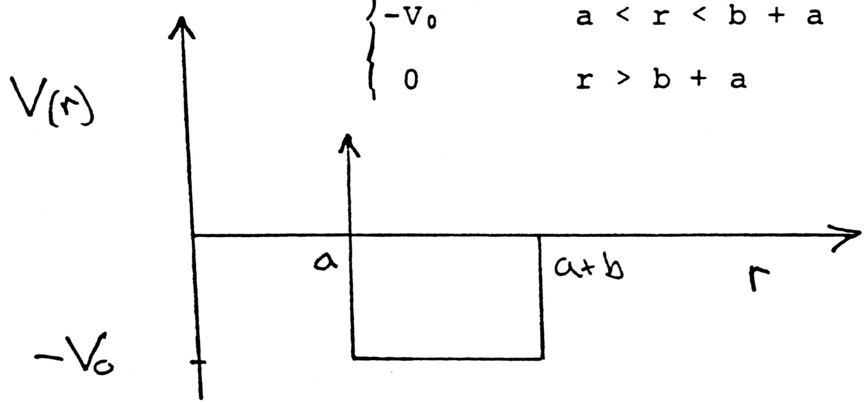
IV.2
Quantum Mechanics

#2: Scattering from a square well with a hard core.

Consider a particle of mass M scattering from a potential V(r)

$$V(r) = \begin{cases} \infty & r < a \\ -V_0 & a < r < b + a \\ 0 & r > b + a \end{cases}$$



At low energies the s wave phase shift can be expanded

$$k \cot \delta \approx -\frac{1}{a_0} + \frac{1}{2} r_0 k^2$$

where k is the particle's momentum.

Assume $\hbar = 1$ and

$$\sqrt{2MV_0}\, b = \pi$$

and calculate the s wave scattering length a_0 and effective range r_0. Show that they are

$$a_0 = a + b$$

$$r_0 = \frac{2}{3}(a + b) - \frac{b^3}{\pi^2 (a + b)^2}$$